THE UNLIKELY INTERSECTION THEORY AND THE COSMETIC SURGERY CONJECTURE

BOGWANG JEON

Abstract. Let $\mathcal{M}$ be a 1-cusped hyperbolic 3-manifold whose cusp shape is not quadratic, and $\mathcal{M}(p/q)$ be its p/q-Dehn filled manifold. We show that if $p/q \neq p'/q'$ with sufficiently large $|p|+|q|$ and $|p'|+|q'|$, there is no orientation preserving isometry between $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$. This resolves the conjecture of C. Gordon, which is so called the Cosmetic Surgery Conjecture, for hyperbolic 3-manifolds belonging to the aforementioned class except for possibly finitely many exceptions for each manifold. We also consider its generalization to more cusped manifolds. The key ingredient of the proof is the unlikely intersection theory developed by E. Bombieri, D. Masser and U. Zannier.

1. Introduction

1.1. **Main Results** Dehn filling is one of the most fundamental topological operations in the field of low dimensional topology. More than 50 years ago, W. Lickorish and A. Wallace showed that any closed connected orientable 3-manifold can be obtained by a Dehn filling on a link complement. In the late 1970's, W. Thurston, as a part of his revolutionary work, showed Dehn filling behaves very nicely under hyperbolic structure by proving that if the original 3-manifold is hyperbolic, then almost all of its Dehn fillings are also hyperbolic. Since then, understanding hyperbolic Dehn filling has become a central topic in the study of 3-dimensional geometry and topology.

However many quantitative questions regarding Dehn filling are still unanswered even for simple cases. For instance the following conjecture, which was proposed by C. Gordon in 1990 [7] (see also Kirby's problem list [11]), is one of the basic questions in the topic, but the complete answer is unknown (see [2], [12], [15], [16], [18] for partial results):

Conjecture 1 (Cosmetic Surgery Conjecture (Hyperbolic Case))**.** *Let $\mathcal{M}$ be a 1-cusped hyperbolic 3-manifold. Let $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$ be the p/q and p'/q'-Dehn filled manifolds of it (respectively) which are also hyperbolic. If*

$$p/q \neq p'/q',$$

then there is no orientation preserving isometry between $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$.

The study of unlikely intersections was first initiated by E. Bombieri, D. Masser, and U. Zannier in the 1990's and it has grown and become an active research area in number theory nowadays [19]. Recently it turned out that results in this field could provide powerful tools to understand algebraic invariants of hyperbolic Dehn fillings. For example, using P. Habegger's work, the author proved the following theorem in [9] and [10]:

Theorem 1.1. *Let $\mathcal{M}$ be an k-cusped hyperbolic 3-manifold. Then the height of the Dehn filling point of any hyperbolic Dehn filling of $\mathcal{M}$ is uniformly bounded.*

This leads to the following two corollaries:

Corollary 1.2. *Let $\mathcal{M}$ be an k-cusped hyperbolic 3-manifold. For $D>0$, there are only a finite number of hyperbolic Dehn fillings of $\mathcal{M}$ whose trace field degrees are bounded by D.*

Corollary 1.3. *There are only a finite number of hyperbolic 3 -manifolds of bounded volume and trace field degree.*

Basically the reason that unlikely intersection theory is applicable to the study of hyperbolic Dehn filling is we can interpret this geometric and topological phenomenon as an algebro-geometric one. More precisely, we can view a hyperbolic k-cusped manifold as an k-dimensional algebraic variety, and Dehn filling on it as the intersection between the corresponding variety and an algebraic subgroup whose index is given by the Dehn filling coefficient. Thus unlikely intersection theory provides a natural framework to understand algebraic invariants of hyperbolic Dehn fillings.

In this paper, following along the same lines, we show another application of unlikely intersection theory to a problem of hyperbolic Dehn filling. First, we resolve the Cosmetic Surgery Conjecture for a hyperbolic 1-cusped manifold whose cusp shape is not quadratic except for finitely many exceptions:

Theorem 1.4. *Let $\mathcal{M}$ be a 1-cusped hyperbolic 3-manifold whose cusp shape is non-quadratic. If*

$$p/q \neq p'/q'$$

for sufficiently large $|p|+|q|$ and $|p'|+|q'|$, then there is no orientation preserving isometry between $\mathcal{M}\big(p/q\big)$ and $\mathcal{M}\big(p'/q'\big)$.

The above theorem is a consequence of the following theorem:[1]

Theorem 1.5. *Let $\mathcal{M}$ be a 1-cusped hyperbolic 3-manifold whose cusp shape is non-quadratic. Let $t_{p/q}$ (where $|t_{p/q}|>1$) be the holonomy of the core geodesic of p/q-Dehn filling. For any p/q and p'/q' such that $|p|+|q|$ and $|p'|+|q'|$ are sufficiently large, if*

$$t_{p/q}=t_{p'/q'} \tag{1.1}$$

then $p/q=p'/q'$.

An analogous extension of the above theorem to a more cusped manifold is clearly false. For example, if $\mathcal{M}$ is the Whitehead link complement, then, since it allows the symmetry between two given cusps, $\mathcal{M}(p_1/q_1,p_2/q_2)$ is equal to $\mathcal{M}(p_2/q_2,p_1/q_1)$ for any p_1/q_1 and p_2/q_2.

Let τ_1 and τ_2 be two cusp shapes of a 2-cusped hyperbolic 3-manifold $\mathcal{M}$. If there is a symmetry between two cusps, then it allows the following relation between τ_1 and τ_2:

$$\tau_1=\frac{a\tau_2+b}{c\tau_2+d}$$

where $a,b,c,d\in\mathbb{Z}$ and $ad-bc=\pm1$. Thus having no symmetry between two cusps can be rephrased algebraically as follows:

$$1,\tau_1,\tau_2,\tau_1\tau_2 \quad \text{are linearly independent over} \quad \mathbb{Q}. \tag{1.2}$$

When τ_1,τ_2 satisfies the condition (1.2), we say $\mathcal{M}$ has rationally independent cusp shapes. Under this hypothesis, we can extend the result of Theorem 1.5 to 2-cusped manifolds as the following theorems show:

[1]If the Dehn filling coefficients are sufficiently large, then the core geodesic is the shortest geodesic for each Dehn filled manifold. Thus, if there exists an isometry between $\mathcal{M}_{(p/q)}$ and $\mathcal{M}_{(p'/q')}$, the holonomy of $\mathcal{M}_{(p/q)}$, which represents the core geodesic of $\mathcal{M}_{(p/q)}$, maps to the holonomy of $\mathcal{M}_{(p'/q')}$.

Theorem 1.6. *Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold having non-quadratic, rationally independent cusp shapes and whose two cusps are Strongly Geometrically Isolated (SGI). If*

$$\big(p_1/q_1, p_2/q_2\big) \neq \big(p'_1/q'_1, p'_2/q'_2\big)$$

for sufficiently large $|p_i|+|q_i|$ and $|p'_i|+|q'_i|$ $(1 \leq i \leq 2)$, then there is no orientation preserving isometry between $\mathcal{M}\big(p_1/q_1, p_2/q_2\big)$ and $\mathcal{M}\big(p'_1/q'_1, p'_2/q'_2\big)$.

The above theorem is an immediate consequence of the following theorem:

Theorem 1.7. *Let $\mathcal{M}$ be the same manifold given in Theorem 1.6. Let*

$$\Big\{t^1_{(p_1/q_1,p_2/q_2)}, t^2_{(p_1/q_1,p_2/q_2)}\Big\}$$

where $|t^i_{(p_1/q_1,p_2/q_2)}| > 1$ be the set of holonomies corresponding to two core geodesics of $\big(p_1/q_1, p_2/q_2\big)$-Dehn filling. If

$$\Big\{t^1_{(p_1/q_1,p_2/q_2)}, t^2_{(p_1/q_1,p_2/q_2)}\Big\} = \Big\{t^1_{(p'_1/q'_1,p'_2/q'_2)}, t^2_{(p'_1/q'_1,p'_2/q'_2)}\Big\}$$

for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ $(1 \leq i \leq 2)$, then

$$\big(p_1/q_1, p_2/q_2\big) = \big(p'_1/q'_1, p'_2/q'_2\big).$$

In other words, the set of holonomies uniquely determines Dehn filling coefficient.

If two cusps of $\mathcal{M}$ are not SGI, its Neumann-Zagier potential function $\Phi(u_1, u_2)$ contains a nonzero term of the form $u_1^i u_2^j$ where $i, j > 0$. In this case, under one mild condition, we prove the following theorem:

Theorem 1.8. *Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold having non-quadratic and rationally independent cusp shapes. Let*

$$\Phi(u_1, u_2) = \tau_1 u_1^2 + \tau_2 u_2^2 + m_{40}u_1^4 + m_{22}u_1^2u_2^2 + m_{04}u_2^4 + \cdots$$

be its Neumann-Zagier potential function such that $m_{22} \neq 0$. If

$$\big(p_1/q_1, p_2/q_2\big) \neq \big(p'_1/q'_1, p'_2/q'_2\big)$$

for sufficiently large $|p_i|+|q_i|$ and $|p'_i|+|q'_i|$ $(1 \leq i \leq 2)$, then there is no orientation preserving isometry between $\mathcal{M}\big(p_1/q_1, p_2/q_2\big)$ and $\mathcal{M}\big(p'_1/q'_1, p'_2/q'_2\big)$.

This is a consequence of the following theorem:

Theorem 1.9. *Let $\mathcal{M}$ be the same manifold given in Theorem 1.8. Let*

$$\Big\{t^1_{(p_1/q_1,p_2/q_2)}, t^2_{(p_1/q_1,p_2/q_2)}\Big\}$$

where $|t^i_{(p_1/q_1,p_2/q_2)}| > 1$ be the set of holonomies corresponding to two core geodesics of $\big(p_1/q_1, p_2/q_2\big)$-Dehn filling. If

$$\Big\{t^1_{(p_1/q_1,p_2/q_2)}, t^2_{(p_1/q_1,p_2/q_2)}\Big\} = \Big\{t^1_{(p'_1/q'_1,p'_2/q'_2)}, t^2_{(p'_1/q'_1,p'_2/q'_2)}\Big\}$$

for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ $(1 \leq i \leq 2)$, then

$$\big(p_1/q_1, p_2/q_2\big) = \big(p'_1/q'_1, p'_2/q'_2\big).$$

In Theorem 1.9, although the statement is conditional upon a coefficient of a Neumann-Zagier potential function, we believe that this condition is a generic case.

1.2. **Key Idea** Let $\mathcal{H}_k$ be the set of algebraic subgroups of co-dimension k in $\left(\overline{\mathbb{Q}^*}\right)^n$. The following theorem was proved in [8]. See Section 3.2 for the definition of $\mathcal{X}^{oa}$.

Theorem 1.10 (P. Habegger)**.** *Let $\mathcal{X} \in \left(\overline{\mathbb{Q}^*}\right)^n$ be a k-dimensional algebraic variety. Then the height of $\mathcal{X}^{oa} \cap \mathcal{H}_k$ is uniformly bounded.*

For algebraic subgroups whose co-dimension is bigger than the dimension of a given variety, we have the following theorem of E. Bombieri, D. Masser and U. Zannier proved in [3].

Theorem 1.11 (E. Bombieri, D. Masser, U. Zannier)**.** *Let $\mathcal{X} \in \left(\overline{\mathbb{Q}^*}\right)^n$ be a variety of dimension k. Then, for any $B \geq 0$, there exists D_B depending only on B such that the degree of P in $\mathcal{X}^{oa} \cap \mathcal{H}_{k+1}$ with $h(P) \leq B$ is bounded by D_B.*

By Northcott's theorem, the above two theorems imply the following:

Theorem 1.12. *Let $\mathcal{X} \in \left(\overline{\mathbb{Q}^*}\right)^n$ be a variety of dimension k. Then $\mathcal{X}^{oa} \cap \mathcal{H}_{k+1}$ is a finite set.*

Now let $\mathcal{M}$ be a hyperbolic 1-cusped manifold and $\mathcal{X}$ be its holonomy variety (see Section 2.2 for the definition of a holonomy variety). Using M, L as the parameters for the holonomies of the longitude-meridian pair, we identify $\mathcal{M}(p/q)$ as an intersection point between $\mathcal{X}$ and an algebraic group defined by $M^pL^q = 1$. If $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$ are isometric (with sufficiently large $|p| + |q|$ and $|p'| + |q'|$), the core geodesic of $\mathcal{M}(p/q)$ maps to the core geodesic of $\mathcal{M}(p'/q')$ under this isometry. Now we consider $\mathcal{X} \times \mathcal{X}$, and associate $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$ to the first and second coordinates of it respectively. We also use M', L' to represent the holonomies of the longitude-meridian pair of the second coordinate. Since the holonomy of the core geodesic of $\mathcal{M}(p/q)$ (resp. $\mathcal{M}(p'/q')$) is M^rL^s (resp. $(M')^{r'}(L')^{s'}$) where $pr - qs = 1$ (resp. $p'r' - q's' = 1$), an isometry between $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$ can be interpreted as an intersection point between $\mathcal{X} \times \mathcal{X}$ and the algebraic group defined by

$$M^pL^q = 1,$$
$$(M')^{p'}(L')^{q'} = 1,$$
$$M^rL^s = (M')^{r'}(L')^{s'}.$$

In other words, the existence of an isometry between two Dehn fillings of different coefficients guarantees the existence of an intersection point between an algebraic variety of dimension 2 and an algebraic subgroup of co-dimension 3. However, in view of Theorem 1.12, there exist at most finitely many such points. Thus, except for those finitely many exceptions, we expect that there exists no isometry between two Dehn filled manifolds having different filling coefficients.

This heuristic argument ignored the difference between $\mathcal{X} \times \mathcal{X}$ and $(\mathcal{X} \times \mathcal{X})^{oa}$ in applying Theorem 1.12, but the techniques and ideas in [3] perfectly work well to achieve our goal.

1.3. **Outline of the paper** In Sections 2 and 3, we study some necessary background on hyperbolic geometry and number theory respectively. In Section 4, we prove some preliminary lemmas which play key roles in the proofs of the main theorems. Finally, in Sections 5, 6, and 7, we establish Theorems 1.5, 1.7, and 1.9 respectively.

1.4. **Acknowledgments** I would like to thank Walter Neumann and Nathan Dunfield for helpful conversations and correspondence. I also thank Abhijit Champanerkar for many useful comments on the paper.

2. Background I (Hyperbolic Geometry)

2.1. **Gluing variety** In this section, we follow the same scheme in [14]. Suppose that $\mathcal{M}$ is a k-cusped manifold whose hyperbolic structure is realized as a union of n geometric tetrahedra having modulus z_v $(1 \leq v \leq n)$. Then the gluing variety of $\mathcal{M}$ is defined by the following form of n equations where each represents the gluing condition at each edge of a tetrahedron:

$$\prod_{v=1}^{n} z_v^{\theta_1(r,v)} \cdot (1-z_v)^{\theta_2(r,v)} = \pm 1 \tag{2.1}$$

for $1 \leq r \leq n$, $\theta_1(r,v), \theta_2(r,v) \in \mathbb{Z}$. It is known that there is redundancy in the above equations so that exactly $n-k$ of them are independent [14]. We denote the solution set of the above equations in $\big(\mathbb{C}\backslash\{0,1\}\big)^n$ by $\mathrm{Hol}(\mathcal{M})$ and the point corresponding to the complete structure by $z^0 = (z_1^0, \dots, z_n^0) \in \mathrm{Hol}(\mathcal{M})$.

Let T_i be a torus cross-section of the i^{th}-cusp and l_i, m_i be the chosen longitude-meridian pair of T_i $(1 \leq i \leq k)$. For each $z \in \mathrm{Hol}(\mathcal{M})$, by giving similarity structures on the tori T_i, the dilation components of the holonomies (of the similarity structures) of l_i and m_i are represented in the following forms:

$$\begin{aligned} \delta(l_i)(z) &= \pm \prod_{v=1}^{n} z_v^{\lambda_1(i,v)} \cdot (1-z_v)^{\lambda_2(i,v)} \\ \delta(m_i)(z) &= \pm \prod_{v=1}^{n} z_v^{\mu_1(i,v)} \cdot (1-z_v)^{\mu_2(i,v)}. \end{aligned} \tag{2.2}$$

Then $\delta(l_i)(z)$ and $\delta(m_i)(z)$ behave very nicely near z^0 [14].

Theorem 2.1. *For each i $(1 \leq i \leq k)$, $\delta(l_i)(z) = 1$ and $\delta(m_i)(z) = 1$ are equivalent in a small neighborhood of z^0.*

Theorem 2.2. *z^0 is a smooth point of Hol($\mathcal{M}$) and the unique point near z^0 with all $\delta(l_i)(z) = 1$ $(1 \leq i \leq k)$.*

By taking logarithms locally near the point z^0, equation (2.1) can be re-written as follows:

$$\sum_{v=1}^{n} \Big(\theta_1(r,v) \cdot \log(z_v) + \theta_2(r,v) \cdot \log(1-z_v)\Big) = c(r) \quad (1 \leq r \leq n-k) \tag{2.3}$$

where $c(r)$ are some suitable constants. In the same way, if we let

$$u_i(z) = \log\big(\delta(l_i)(z)\big) \quad (1 \leq i \leq k) \tag{2.4}$$

$$v_i(z) = \log\big(\delta(m_i)(z)\big) \quad (1 \leq i \leq k) \tag{2.5}$$

in a small neighborhood of z^0, then $v_1, \dots, v_k$ are parametrized holomorphically in terms of $u_1, \dots, u_k$ as below [14].

Theorem 2.3. *In a neighborhood of the origin in $\mathbb{C}^k$ (with $u_1, \dots, u_k$ as coordinates), the following holds for each i $(1 \leq i \leq k)$:*

(1) $v_i = u_i \cdot \tau_i(u_1, \dots, u_k)$ where $\tau_i(u_1, \dots, u_k)$ is an even holomorphic function of its arguments with $\tau_i(0, \dots, 0) = \tau_i$ $(1 \leq i \leq k)$.

(2) There is an holomorphic function $\Phi(u_1, \dots, u_k)$ such that $v_i = \dfrac{1}{2}\dfrac{\partial \Phi}{\partial u_i}$ $(1 \leq i \leq k)$ and

$\Phi(0,\dots,0)=0$.
(3) $\Phi(u_1,\dots,u_k)$ *is even in each argument and it has Taylor expansion of the following form:*

$$\Phi(u_1,\dots,u_k)=\tau_1u_1^2+\cdots+\tau_ku_k^2+(\text{higher order}).$$

We call τ_i the cusp shape of T_i with respect to l_i,m_i and $\Phi(u_1,\dots,u_k)$ the *Neumann-Zagier potential function* of $\mathcal{M}$ with respect to $\{l_i,m_i\}_{1\leq i\leq k}$.

Considering u_i,v_i $(1\leq i\leq k)$ as coordinates, we denote the complex manifold, which is defined locally near $(0,\dots,0)$ in $\mathbb{C}^{2k}$ via the following holomorphic functions

$$v_i=u_i\cdot\tau_i(u_1,\dots,u_k)\quad(1\leq i\leq k)\tag{2.6}$$

by $\text{Def}(\mathcal{M})$. Note that $\text{Def}(\mathcal{M})$ is locally biholomorphic to a small neighborhood of z_0 in $\text{Hom}(\mathcal{M})$.

2.2. **Holonomy variety (Deformation variety)**

Thinking of the holonomies of the meridian-longitude pairs $\delta(l_j)(z)$ and $\delta(m_j)(z)$ $(1\leq j\leq k)$ in (2.2) as new variables, we consider the variety defined by the following equations in $\mathbb{C}^{n+2k}$:

$$\begin{aligned}\prod_{v=1}^{n}z_v^{\theta_1(r,v)}\cdot(1-z_v)^{\theta_2(r,v)}&=\pm1\\ L_i=\pm\prod_{v=1}^{n}z_v^{\lambda_1(i,v)}&\cdot(1-z_v)^{\lambda_2(i,v)}\\ M_i=\pm\prod_{v=1}^{n}z_v^{\mu_1(i,v)}&\cdot(1-z_v)^{\mu_2(i,v)}\end{aligned}\tag{2.7}$$

where $1\leq r\leq n-k$ and $1\leq i\leq k$. We define this as the holonomy variety of $\mathcal{M}$ and denote it by $\mathcal{X}$. Then the point corresponding to the complete structure is

$$(z_1^0,\dots,z_n^0,1,\dots,1)\tag{2.8}$$

and it is also a smooth point of $\mathcal{X}$. (So $\text{Def}(\mathcal{M})$ is biholomorphic to a neighborhood of z^0 in $\mathcal{X}$ as well.) By abusing the notation, we still denote (2.8) by z^0. Throughout the paper, we will only be interested in a small neighborhood of z^0 (see Theorem 2.4 and Theorem 2.5). Also, in the proofs of the main theorems, we will usually work with $\text{Def}(\mathcal{M})$ instead of $\mathcal{X}$ since $\text{Def}(\mathcal{M})$ is easier to deal with. For instance, if H is an algebraic variety defined by

$$\begin{gathered}M_1^{a_{11}}L_1^{b_{11}}\cdots M_k^{a_{1k}}M_k^{b_{1k}}=1,\\ \cdots\\ M_1^{a_{n1}}L_1^{b_{n1}}\cdots M_k^{a_{nk}}M_k^{b_{nk}}=1,\end{gathered}$$

then it is biholomorphic to the linear space defined by

$$\begin{gathered}a_{11}u_1+b_{11}v_1+\cdots+a_{1k}u_k+b_{1k}v_k=0,\\ \cdots\\ a_{n1}u_1+b_{n1}v_1+\cdots+a_{nk}u_k+b_{nk}v_k=0,\end{gathered}$$

and so $\mathcal{X}\cap H$ is locally biholomorphic (near z^0) to the complex manifold defined by the following holomorphic equations:

$$\begin{aligned} a_{11}u_1+b_{11}u_1\tau_1(u_1,\dots,u_k)+\cdots+a_{1k}u_k+b_{1k}u_k\tau_k(u_1,\dots,u_k)&=0,\\ \cdots\\ a_{n1}u_1+b_{n1}u_1\tau_1(u_1,\dots,u_k)+\cdots+a_{nk}u_k+b_{nk}u_k\tau_k(u_1,\dots,u_k)&=0. \end{aligned} \tag{2.9}$$

Thus we can get the information about the dimension of $\mathcal{X}\cap H$ by analyzing the rank of the Jacobian of (2.9) using Theorem 2.3.

Remark. For a given cusped hyperbolic 3-manifold, its gluing variety (holonomy variety and potential function as well) depends on a choice of meridian-longitude pair, but they are all isomorphic. Sometimes changing basis from one to another and working with a different (isomorphic) variety are quite useful. We will use this technique in the proof of Theorem 1.9 in the last section.

2.3. **Dehn Filling** Hyperbolic Dehn filling can be defined in a few slightly different ways. In this paper, we adopt the definition that, after attaching a new torus, the core of the torus is always isotopic to a geodesic of the Dehn filled manifold.

Let

$$\mathcal{M}(p_1/q_1,...,p_k/q_k)$$

be the $(p_1/q_1,\dots,p_k/q_k)$-Dehn filled manifold of $\mathcal{M}$. By the Seifert-Van Kampen theorem, the fundamental group of $\mathcal{M}(p_1/q_1,...,p_k/q_k)$ is obtained by adding the relations

$$m_1^{p_1}l_1^{q_1}=1,\quad ...\quad, m_k^{p_k}l_k^{q_k}=1$$

to the fundamental group of $\mathcal{M}$. Hence, on the holonomy variety of $\mathcal{M}$, the hyperbolic structure of $\mathcal{M}(p_1/q_1,...,p_k/q_k)$ is identified with a point satisfying the additional equations corresponding to the above relations. More precisely, if the holonomy variety $\mathcal{X}$ of $\mathcal{M}$ is defined by

$$f_i(z_1,\dots,z_n,M_1,\dots,M_k,L_1,\dots,L_k)=0\quad(1\le i\le s), \tag{2.10}$$

then a holonomy representation of $\mathcal{M}$ which gives rise to an incomplete structure inducing $\mathcal{M}(p_1/q_1,...,p_k/q_k)$ is a point on $\mathcal{X}$ satisfying the following equations:

$$M_1^{p_1}L_1^{q_1}=1,\quad ...\quad, M_k^{p_k}L_k^{q_k}=1. \tag{2.11}$$

We call (2.11) the *Dehn filling equations* with coefficient $(p_1/q_1,\dots,p_k/q_k)$ and the points inducing the hyperbolic structure on $\mathcal{M}(p_1/q_1,...,p_k/q_k)$ the *Dehn filling points* corresponding to $\mathcal{M}(p_1/q_1,...,p_k/q_k)$. Let

$$(M_1,L_1,\dots,M_k,L_k)=(t_1^{-q_1},t_1^{p_1},\dots,t_k^{-q_k},t_k^{p_k}) \tag{2.12}$$

be the last $(2k)$-coordinates of one of the Dehn filling points corresponding to $\mathcal{M}(p_1/q_1,...,p_k/q_k)$ such that $|t_i|>1$ $(1\le i\le k)$. Then the holonomy of each core geodesic $m_i^{s_i}l_i^{r_i}$ where $p_ir_i-q_is_i=1$ is

$$M_i^{s_i}L_i^{r_i}=(t_i^{-q_i})^{s_i}(t_i^{p_i})^{r_i}=t_i^{-q_is_i+p_is_i}=t_i.$$

We define

$$\{t_1,\dots,t_k\}$$

as the set of *holonomies* of the Dehn filling coefficient $(p_1/q_1,...,p_k/q_k)$.

The following theorems are parts of Thurston's hyperbolic Dehn filling theory [14][17].

Theorem 2.4. *Using the same notation as above,*

$$(t_1^{-q_1}, t_1^{p_1}, \dots, t_k^{-q_k}, t_k^{p_k})$$

converges to $(1, \dots, 1)$ *as* $|p_i| + |q_i|$ *goes to* ∞ *for* $1 \leq i \leq k$.

Theorem 2.5. *Using the same notation as above,*

$$(t_1, \dots, t_k)$$

converges to $(1, \dots, 1)$ *as* $|p_i| + |q_i|$ *goes to* ∞ *for* $1 \leq i \leq k$.

3. Background II (Number Theory)

3.1. **Height** The height $h(\alpha)$ of an algebraic number α is defined as follows:

Definition 3.1. *Let* K *be an any number field containing* α, V_K *be the set of places of* K, *and* $K_v, \mathbb{Q}_v$ *be the completions at* $v \in V_K$. *Then*

$$h(\alpha) = \sum_{v \in M_K} \log \big(\max\{1, |\alpha|_v\}\big)^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]}.$$

Note that the above definition does not depend on the choice of K. That is, for any number field K containing α, it gives us the same value.

The first property in the following theorem is a classical result due to D. Northcott and the rest can be easily deduced from the definition [6].

Theorem 3.2. *(1) There are only finitely many algebraic numbers of bounded height and degree.*
(2) $h(\alpha^n) = |n| h(1/\alpha)$ *for* $\alpha \in \overline{\mathbb{Q}}$.
(3) $h(\alpha_1 + \cdots + \alpha_r) \leq \log r + h(\alpha_1) + \cdots + h(\alpha_r)$ *for* $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}$.
(4) $h(\alpha_1 \cdots \alpha_r) \leq h(\alpha_1) + \cdots + h(\alpha_r)$ *for* $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}$.

If $\alpha = (\alpha_1, ..., \alpha_n) \in \overline{\mathbb{Q}}^n$ is an n-tuple of algebraic numbers, the definition can be generalized as follows:

Definition 3.3. *Let* K *be an any number field containing* $\alpha_1, ..., \alpha_n$, V_K *be the set of places of* K, *and* $K_v, \mathbb{Q}_v$ *be the completions at* v. *Then*

$$h(\alpha) = \sum_{v \in V_K} \log \big(\max\{1, |\alpha_1|_v, ..., |\alpha_n|_v\}\big)^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]}.$$

Similar to Theorem 3.2, the following inequalities hold:

$$\max\{h(\alpha_1), \dots, h(\alpha_n)\} \leq h(\alpha) \leq h(\alpha_1) + \cdots + h(\alpha_n). \tag{3.1}$$

3.2. **Anomalous Subvarieties** In this section, we identify G_m^n with the non-vanishing of the coordinates $x_1, \dots, x_n$ in the affine n-space $\overline{\mathbb{Q}}^n$ or $\mathbb{C}^n$ (i.e. $G_m^n = (\overline{\mathbb{Q}}^*)^n$ or $(\mathbb{C}^*)^n$). An algebraic subgroup H_Λ of G_m^n is defined as the set of solutions satisfying equations $x_1^{a_1} \cdots x_n^{a_n} = 1$ where the vector $(a_1, \dots, a_n)$ runs through a lattice $\Lambda \subset \mathbb{Z}^n$. If Λ is primitive, then we call H_Λ an irreducible algebraic subgroup or algebraic torus. By a coset K, we mean a translate gH of some algebraic subgroup H by some $g \in G_m^n$. For more properties of algebraic subgroups and G_m^n, see [6].

Definition 3.4. *An irreducible subvariety $\mathcal{Y}$ of $\mathcal{X}$ is anomalous (or better, $\mathcal{X}$-anomalous) if it has positive dimension and lies in a coset K in G_m^n satisfying*

$$\dim K \leq n - \dim \mathcal{X} + \dim \mathcal{Y} - 1.$$

The quantity $\dim \mathcal{X} + \dim K - n$ is what one would expect for the dimension of $\mathcal{X} \cap K$ when $\mathcal{X}$ and K were in general position. Thus we can understand anomalous subvarieties of $\mathcal{X}$ as the ones that are unnaturally large intersections with cosets of algebraic subgroups of G_m^n.[2]

Definition 3.5. *The deprived set $\mathcal{X}^{oa}$ is what remains of $\mathcal{X}$ after removing all anomalous subvarieties.*

Definition 3.6. *An anomalous subvariety of $\mathcal{X}$ is maximal if it is not contained in a strictly larger anomalous subvariety of $\mathcal{X}$.*

The following theorem tells us the structure of anomalous subvarieties (Theorem 1 of [4]).

Theorem 3.7. *Let $\mathcal{X}$ be an irreducible variety in G_m^n of positive dimension defined over $\overline{\mathbb{Q}}$.*
(a) For any torus H with

$$1 \leq h = n - \dim H \leq \dim \mathcal{X}, \tag{3.2}$$

the union Z_H of all subvarieties $\mathcal{Y}$ of $\mathcal{X}$ contained in any coset K of H with

$$\dim \mathcal{Y} = \dim \mathcal{X} - h + 1 \tag{3.3}$$

is a closed subset of $\mathcal{X}$, and the product HZ_H is not Zariski dense in G_m^n.
(b) There is a finite collection $\Psi = \Psi_{\mathcal{X}}$ of such tori H such that every maximal anomalous subvariety $\mathcal{Y}$ of $\mathcal{X}$ is a component of $\mathcal{X} \cap gH$ for some H in Ψ satisfying (3.2) *and* (3.3) *and some g in Z_H. Moreover $\mathcal{X}^{oa}$ is obtained from $\mathcal{X}$ by removing the Z_H of all H in Ψ, and thus it is open in $\mathcal{X}$ with respect to the Zariski topology.*

Following the definition given in [5], we have a refined version of Definition 3.4 as follows:

Definition 3.8. *We say that an irreducible subvariety $\mathcal{Y}$ of $\mathcal{X}$ is* b-anomalous *if it has positive dimension and lies in some coset of dimension*

$$n - (b + \dim \mathcal{X}) + \dim \mathcal{Y}.$$

The following strengthening version of Theorem 3.7 can be also found in [5]:

Theorem 3.9. *Suppose that $\mathcal{X}$ is an irreducible variety in G_m^n defined over $\overline{\mathbb{Q}}$. Then any b-anomalous subvariety $\mathcal{Y}$ of $\mathcal{X}$ lies in a coset $gH^{(0)}$ ($g \in G_m^n$) satisfying*

$$\dim H^{(0)} = n - (b + \dim \mathcal{X}) + \dim \mathcal{Y}, \qquad \deg H^{(0)} \leq D \tag{3.4}$$

where D depends only on n and $\deg \mathcal{X}$*.*

Now we state the bounded height theorem due to P. Habegger.

Theorem 3.10. [8] *Let $\mathcal{X} \subset G_m^n$ be an irreducible variety over $\overline{\mathbb{Q}}$. The height is bounded in the intersection of $\mathcal{X}^{oa}$ with the union of algebraic subgroups of dimension $\leq n - \dim \mathcal{X}$.*

As we mentioned earlier, using the proof of the above theorem, the author proved the following theorem in [9] and [10]:

[2] By Theorem 2.1, for each i ($1 \leq i \leq k$), $M_i = 1, L_i = 1$ produces an anomalous subvariety by intersecting the holonomy variety $\mathcal{X}$.

Theorem 3.11. *M be a k-cusped hyperbolic 3-manifold. Then the height of any Dehn filling point is uniformly bounded.*

Lastly, we mention a couple of lemmas and a theorem which are crucial in the proofs of the main theorems.

Definition 3.12. *We say $\eta_1, \dots, \eta_r \in \overline{\mathbb{Q}}$ are multiplicatively dependent if there exist $a_1, \dots, a_r \in \mathbb{Z}$ such that*

$$\eta_1^{a_1} \cdots \eta_r^{a_r} = 1. \tag{3.5}$$

Otherwise, we say they are multiplicatively independent.

The following theorem is Lemma 7.1 in [3].

Lemma 3.13. [3] *Given $r \geq 1$ there are positive constants $c(r)$ and $k(r)$ with the following property. Let K be a cyclotomic extension of degree d over $\mathbb{Q}$ and let $\eta_1, \dots, \eta_r$ be multiplicatively independent non-zero algebraic numbers with $[K(\eta_1, \dots, \eta_r) : K] = d$. Then*

$$h(\eta_1) \cdots h(\eta_r) \geq \frac{c(r)}{\tilde{d}(\log(3d\tilde{d})^{k(r)}}.$$

Lemma 3.14 (Siegel)**.** *Consider the following linear equations:*

$$\sum_{i=1}^{n} a_{ij} X_i \quad (1 \leq j \leq r). \tag{3.6}$$

Let $\mathbf{v_j} = (a_{1j}, \dots, a_{nj})$ $(1 \leq j \leq r)$ and $\prod = |\mathbf{v_1}| \cdots |\mathbf{v_r}|$ where $|\mathbf{v_j}|$ represents its Euclidean length. Then there exist an universal constant $c > 0$ and $(n-r)$-independent vectors $\mathbf{b_i} \in \mathbb{Z}^n$ $(1 \leq i \leq n-r)$ which vanish at the forms in (3.6) and satisfy

$$|\mathbf{b_1}| \leq \cdots \leq |\mathbf{b_{n-r}}|, \quad |\mathbf{b_1}| \cdots |\mathbf{b_{n-r}}| \leq c \prod.$$

See [6] for a proof of the above theorem. The following theorem was proved in [5].

Theorem 3.15. [5] *Let $\mathcal{C}$ be a complex algebraic curve and $\mathcal{H}_2$ be the set of all the algebraic subgroups of co-dimension 2. If $\mathcal{H}_2 \cap \mathcal{C}$ is not finite, then $\mathcal{C}$ is contained in an algebraic subgroup.*

3.3. **Strong Geometric Isolation and Anomalous Subvarieties** Strong geometric isolation was first introduced by W. Neumann and A. Reid in [13]. Geometrically, it simply means that one subset of cusps moves independently without affecting the rest. Using Theorem 4.3 in [13], we give one of the equivalent forms of the definition as follows:

Definition 3.16. *Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold. We say two cusps of $\mathcal{M}$ are strongly geometrically isolated (SGI) if v_1 only depends on u_1 and v_2 only depends on v_1.*

The following theorems were proved in [9].

Theorem 3.17. *Let $\mathcal{M}$ be a 2-cusped manifold with rationally independent cusp shapes and $\mathcal{X}$ be its holonomy variety. Then a maximal anomalous subvariety of $\mathcal{X}$ containing z^0 is either contained in $M_1 = L_1 = 1$ or $M_2 = L_2 = 1$.*

Theorem 3.18. *Let $\mathcal{M}$ is a 2-cusped hyperbolic 3-manifold having rationally independent cusp shapes. If $\mathcal{X}^{oa} = \emptyset$, then two cusps of $\mathcal{M}$ are SGI.*

4. Preliminary Lemmas

Lemma 4.1. *Let*

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix} \tag{4.1}$$

be an integer matrix of rank 2 *and* $(p,q),(p',q')$ *be two co-prime pairs such that*

$$\begin{aligned} -qa_1 + pb_1 - q'c_1 + p'd_1 = 0, \\ -qa_2 + pb_2 - q'c_2 + p'd_2 = 0. \end{aligned} \tag{4.2}$$

For a non-quadratic number τ, *if the rank of the following* (2×2)*-matrix*

$$\begin{pmatrix} a_1 + b_1\tau & c_1 + d_1\tau \\ a_2 + b_2\tau & c_2 + d_2\tau \end{pmatrix} \tag{4.3}$$

is equal to 1, *then we have either* $(p,q) = (p',q')$ *or* $(p,q) = (-p',-q')$. *In particular, if* $(p,q) = (p',q')$, *then* (4.1) *is of the form*

$$\begin{pmatrix} a_1 & b_1 & a_1 & b_1 \\ a_2 & b_2 & a_2 & b_2 \end{pmatrix},$$

and if $(p,q) = (-p',-q')$, *then* (4.1) *is of the form*

$$\begin{pmatrix} a_1 & b_1 & -a_1 & -b_1 \\ a_2 & b_2 & -a_2 & -b_2 \end{pmatrix}.$$

Proof. Since the rank of (4.3) is 1, we have

$$(a_1 + b_1\tau)(c_2 + d_2\tau) = (a_2 + b_2\tau)(c_1 + d_1\tau).$$

By the assumption, τ is not quadratic, so we get

$$a_1c_2 = a_2c_1, \tag{4.4}$$

$$b_1d_2 = b_2d_1, \tag{4.5}$$

$$b_1c_2 + a_1d_2 = a_2d_1 + b_2c_1. \tag{4.6}$$

(1) If none of a_i, b_i, c_i, d_i $(i = 1,2)$ are zero, then there exist $m,n \in \mathbb{Q}$ such that

$$\begin{aligned} (a_1,a_2) = m(c_1,c_2), \\ (b_1,b_2) = n(d_1,d_2). \end{aligned} \tag{4.7}$$

By (4.6), we get

$$nd_1c_2 + mc_1d_2 = mc_2d_1 + nd_2c_1,$$

which is equivalent to

$$(n-m)(d_1c_2 - d_2c_1) = 0.$$

(a) If $d_1c_2 - d_2c_1 = 0$, then there exists $l \in \mathbb{Q}$ such that $l(b_1,b_2) = (c_1,c_2)$. Combining with (4.7), we get

$$(a_1,a_2) = m(c_1,c_2) = ml(b_1,b_2) = mln(d_1,d_2).$$

This implies that $(a_1,b_1,c_1,d_1) = t(a_2,b_2,c_2,d_2)$ for some $t \in \mathbb{Q}$, and it contradicts the fact that (4.1) is a matrix of rank 2.

(b) If $m = n$, then, by combining (4.7) with (4.2), we have

$$-qmc_1 + pmd_1 - q'c_1 + p'd_1 = 0,$$
$$-qmc_2 + pmd_2 - q'c_2 + p'd_2 = 0,$$

which is equivalent to

$$c_1(-qm - q') + d_1(pm + p') = 0,$$
$$c_2(-qm - q') + d_2(pm + p') = 0.$$

(i) If $-qm - q' \neq 0$ and $pm + p' \neq 0$, then this implies that $(c_1, c_2) = l(d_1, d_2)$ for some $l \in \mathbb{Q}$ and so

$$(a_1, a_2) = m(c_1, c_2) = mnl(b_1, b_2) = ml(d_1, d_2),$$

which implies $(a_1, b_1, c_1, d_2) = t(a_2, b_2, c_2, d_2)$ for some t. Again, this contradicts the fact that (4.1) is a matrix of rank 2.

(ii) If $-qm - q' = 0$ (resp. $pm + p' = 0$), then $pm + p' = 0$ (resp. $-qm - q' = 0$). Thus $(p', q') = (-mp, -mq)$ and, since (p, q) and (p', q') are co-prime pairs, we get

$$m = 1, \quad (p, q) = -(p', q'), \quad (a_1, a_2) = (c_1, c_2), \quad (b_1, b_2) = (d_1, d_2)$$

or

$$m = -1, \quad (p, q) = (p', q'), \quad (a_1, a_2) = -(c_1, c_2), \quad (b_1, b_2) = -(d_1, d_2).$$

(2) One of a_i, b_i, c_i, d_i is equal to 0. By symmetry, it is enough to consider the case $a_1 = 0$. If $a_1 = 0$, then $a_2 = 0$ or $c_1 = 0$ by (4.4). We consider each case step by step below.

(a) If $a_1 = a_2 = 0$, then, by (4.5) and (4.6), we get

$$b_1 d_2 = b_2 d_1, \quad b_1 c_2 = b_2 c_1. \tag{4.8}$$

(i) If none of b_i, c_i, d_i are zero, then $(b_1, b_2) = m(c_1, c_2)$ and $(b_1, b_2) = n(d_1, d_2)$ for some $n, m \in \mathbb{Q}$. But this contradicts the fact that (4.1) is a matrix of rank 2.

(ii) Suppose $b_1 = 0$ or $b_2 = 0$. Without loss of generality, we assume $b_1 = 0$. By (4.8), if $b_2 \neq 0$, then $c_1 = d_1 = 0$ and so $a_1 = b_1 = c_1 = d_1 = 0$. But this contradicts the fact that (4.1) is a matrix of rank 2. Thus $b_2 = 0$ and the matrix (4.1) is of the following form:

$$\begin{pmatrix} 0 & 0 & c_1 & d_1 \\ 0 & 0 & c_2 & d_2 \end{pmatrix}. \tag{4.9}$$

However, in this case, we have $(c_1, d_1) = m(q', p') = n(c_2, d_2)$ for some $m, n \in \mathbb{Q}$ by (4.2), which, again, contradicts the fact that the rank of (4.9) is 2.

(iii) Suppose $b_1, b_2 \neq 0$ and one of c_i, d_i is zero. Without loss of generality, we consider $c_1 = 0$. Then, by (4.8), $c_2 = 0$. Since $b_1 d_2 = d_1 b_2$ but $b_1, b_2 \neq 0$, it contradicts the fact that the rank of (4.1) is 2.

(b) If $a_1 = c_1 = 0$, then

$$b_1 d_2 = b_2 d_1, \quad b_1 c_2 = a_2 d_1. \tag{4.10}$$

(i) If none of b_i, d_i, c_2, d_2 are zero, then $(b_1, d_1) = m(b_2, d_2) = n(a_2, c_2)$ for some $m, n \in \mathbb{Q}$. By (4.2), we get

$$pb_1 + p'd_1 = 0 \Rightarrow pa_2 + p'c_2 = 0,$$

$$-qa_2 + p\frac{n}{m}a_2 - q'c_2 + p'\frac{n}{m}c_2 = \Big(-q + \frac{pn}{m}\Big)a_2 + \Big(-q' + \frac{p'n}{m}\Big)c_2 = 0,$$

and so

$$(-p', p) = r(a_2, c_2) = s\Big(-q' + \frac{p'n}{m}, q - \frac{pn}{m}\Big) = l(-q', q)$$

for some $r, s, l \in \mathbb{Q}$. This implies $(p, q) = (p', q')$ or $(p, q) = -(p', q')$. It is clear that $b_1 = -d_1,\ a_2 = -c_2,\ b_2 = -d_2$ when $(p, q) = (p', q')$ and $b_1 = d_1,\ a_2 = c_2,\ b_2 = d_2$ when $(p, q) = -(p', q')$.

(ii) If $b_1 = 0$, then $a_1 = b_1 = c_1 = 0$, which contradicts the condition (4.2).

(iii) If $d_1 = 0$, then $a_1 = c_1 = d_1 = 0$, which contradicts the condition (4.2).

(iv) If $b_2 = 0$ (with $b_1 \neq 0$), then $d_2 = 0$ by (4.10). Since we already dealt with the cases $a_2 = 0$ and $d_1 = 0$ above, we assume $a_2 \neq 0$, $d_1 \neq 0$ and so $(b_1, d_1) = m(a_2, c_2)$ for some $m \in \mathbb{Q}$. By (4.2), we have

$$pb_1 + p'd_1 = 0 \Rightarrow pa_2 + p'c_2 = 0,$$
$$-qa_2 - q'c_2 = 0.$$

Thus $(p, p') = n(-c_2, a_2) = l(q, q')$ for some $n, l \in \mathbb{Q}$. This implies $(p, q) = (p', q')$ or $(p, q) = -(p', q')$.

(v) If $d_2 = 0$, then $b_2 = 0$ or $d_1 = 0$ by (4.10). So it falls into the above cases that we already considered.

(vi) If $c_2 = 0$, then $a_2 = 0$ or $d_1 = 0$ by (4.10), which are also the cases that we already discussed above.

□

Lemma 4.2. *Let*

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix} \tag{4.11}$$

be an integer matrix of rank 2*, and* $(p, q), (p', q')$ *be two nonzero pairs (i.e.* $p, q, p', q' \neq 0$*) such that*

$$-qa_1 + pb_1 - q'c_1 + p'd_1 = 0, \tag{4.12}$$
$$-qa_2 + pb_2 - q'c_2 + p'd_2 = 0. \tag{4.13}$$

If τ_1 *and* τ_2 *are two non-real, non-quadratic algebraic numbers which are rationally independent, then the rank of the following matrix*

$$\begin{pmatrix} a_1 + b_1\tau_1 & c_1 + d_1\tau_2 \\ a_2 + b_2\tau_1 & c_2 + d_2\tau_2 \end{pmatrix}$$

is always equal to 2.

Proof. Suppose that the rank is equal to 1. In other words,

$$(a_1 + b_1\tau_1)(c_2 + d_2\tau_2) = (c_1 + d_1\tau_1)(a_2 + b_2\tau_2),$$

and as $1, \tau_1, \tau_2, \tau_1\tau_2$ are linearly independent over $\mathbb{Q}$, this is equivalent to

$$a_1c_2 - c_1a_2 = 0 \tag{4.14}$$
$$b_1c_2 - c_1b_2 = 0 \tag{4.15}$$
$$a_1d_2 - d_1a_2 = 0 \tag{4.16}$$
$$b_1d_2 - d_1b_2 = 0. \tag{4.17}$$

Claim 4.3. *The equations* (4.14)-(4.17) *induce either* $a_1 = a_2 = b_1 = b_2 = 0$ *(with* $c_1d_2 - c_2d_1 \neq 0$*) or* $c_1 = c_2 = d_1 = d_2 = 0$ *(with* $a_1b_2 - a_2b_1 \neq 0$*).*

Proof. If none of a_i, b_i, c_i, d_i $(i = 1, 2)$ are zero, then (4.14)-(4.17) imply the two nonzero vectors (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2) are linearly dependent over $\mathbb{Q}$. But this is impossible because (4.11) is a matrix of rank 2. Without loss of generality, let us assume $a_1 = 0$. Then, by (4.14) and (4.16), we have the following two cases:

(1) $a_2 = 0$
In this case, the problem is reduced to the following:

$$b_1c_2 - c_1b_2 = 0, \tag{4.18}$$
$$b_1d_2 - d_1b_2 = 0, \tag{4.19}$$
$$(b_1, c_1, d_1) \text{ and } (b_2, c_2, d_2) \text{ are linearly independent.} \tag{4.20}$$

Similar to above, if none of b_i, c_i, d_i $(i = 1, 2)$ are zero, then (b_1, c_1, d_1) and (b_2, c_2, d_2) are linearly dependent over $\mathbb{Q}$ by (4.18) and (4.19), contradicting (4.20). So at least one of b_i, c_i, d_i $(i = 1, 2)$ is zero and the situation is divided into the following two subcases.

(a) $b_1 = 0$ or $b_2 = 0$
By symmetry, it is enough to consider the case $b_1 = 0$. If $b_1 = 0$, then $b_2 = 0$ or $c_1 = 0$ (from (4.18)) and $b_2 = 0$ or $d_1 = 0$ (from (4.19)). If $b_2 = 0$, then we get the desired result (i.e. $a_1 = a_2 = b_1 = b_2 = 0$). Otherwise, if $c_1 = d_1 = 0$, this contradicts the fact that (a_1, b_1, c_1, d_1) is a nonzero vector.

(b) $c_1 = 0$ or $c_2 = 0$ or $d_1 = 0$ or $d_2 = 0$ (with $b_1, b_2 \neq 0$)
Here, also by symmetry, it is enough to prove the first case $c_1 = 0$. If $b_1, b_2 \neq 0$ and $c_1 = 0$, then $c_2 = 0$ by (4.18) and the problem is further simplified to the following:

$$b_1d_2 - d_1b_2 = 0,$$
$$(b_1, d_1) \text{ and } (b_2, d_2) \text{ are linearly independent.}$$

However this doesn't hold regardless of the values of d_1 and d_2.

(2) $a_2 \neq 0$ and so $c_1 = d_1 = 0$.
Since (a_1, b_1, c_1, d_1) is a nonzero vector, b_1 is nonzero and $c_2 = d_2 = 0$ by (4.15) and (4.17). As a result, we get $c_1 = c_2 = d_1 = d_2 = 0$, which is the second desired result of the statement.

So Claim 4.3 holds. ∎

Without loss of generality, we suppose that $c_i = d_i = 0$ $(i = 1, 2)$. If none of a_i, b_i are zero, then, by (4.12), we have

$$-qa_1 + pb_1 = 0, \quad -qa_2 + pb_2 = 0$$

and so $(a_1, b_1) = m(p, q) = n(a_2, b_2)$ for some $m, n \in \mathbb{Q}$. But this contradicts the fact that the rank of (4.11) is 2. On the other hand, if one of a_i, b_i is zero, it contradicts the condition (4.12). This completes the proof of the lemma. □

5. 1-cusped case (Proof of Theorem 1.5)

In $\mathcal{X} \times \mathcal{X}$ $(\subset \mathbb{C}^{n+2k} \times \mathbb{C}^{n+2k})$, we use the following coordinates:

$$(z_1, \dots, z_n, M_1, L_1, \dots, M_k, L_k) \times (z_1', \dots, z_n', M_1', L_1', \dots, M_k', L_k').$$

Similarly, we use the following coordinates:

$$(u_1, v_1, \dots, u_k, v_k) \times (u_1', v_1', \dots, u_k', v_k')$$

for $\mathrm{Def}(\mathcal{M}) \times \mathrm{Def}(\mathcal{M})$ $(\subset \mathbb{C}^{2k} \times \mathbb{C}^{2k})$.

Lemma 5.1. *Let M be a 1-cusped hyperbolic 3-manifold and $\mathcal{X}$ be its holonomy variety. Let H be a 2-dimensional algebraic subgroup defined by*

$$M^{a_1} L^{b_1} (M')^{c_1} (L')^{d_1} = 1,$$
$$M^{a_2} L^{b_2} (M')^{c_2} (L')^{d_2} = 1.$$

Suppose that (p, q) and (p', q') are co-prime pairs such that

$$\begin{aligned} -qa_1 + pb_1 - q'c_1 + p'd_1 = 0, \\ -qa_2 + pb_2 - q'c_2 + p'd_2 = 0. \end{aligned} \tag{5.1}$$

If the component of $(\mathcal{X} \times \mathcal{X}) \cap H$ containing z^0 is a 1-dimensional anomalous subvariety of $\mathcal{X} \times \mathcal{X}$, then $(p, q) = (p', q')$ or $(p, q) = -(p', q')$.

Proof. Taking logarithm to each coordinate, H is locally biholomorphic to the linear space defined by the following equations:

$$a_1 u + b_1 v + c_1 u' + d_1 v' = 0, \tag{5.2}$$

$$a_2 u + b_2 v + c_2 u' + d_2 v' = 0. \tag{5.3}$$

Since $\mathrm{Def}(\mathcal{M}) \times \mathrm{Def}(\mathcal{M})$ are parametrized by

$$v = h(u) = \tau u + m u^3 + \cdots, \quad v' = h(u') = \tau u' + m(u')^3 + \cdots,$$

$(\mathcal{X} \times \mathcal{X}) \cap H$ is locally biholomorphic to a complex manifold defined by

$$\begin{aligned} a_1 u + b_1 h(u) + c_1 u' + d_1 h(u') = a_1 u + b_1(\tau u + m u^3 + \cdots) + c_1 u' + d_1(\tau u' + m(u')^3 + \cdots) = 0, \\ a_2 u + b_2 h(u) + c_2 u' + d_2 h(u') = a_2 u + b_2(\tau u + m u^3 + \cdots) + c_2 u' + d_2(\tau u' + m(u')^3 + \cdots) = 0. \end{aligned} \tag{5.4}$$

If $(\mathcal{X} \times \mathcal{X}) \cap H$ contains an 1-dimensional anomalous subvariety containing z^0, then, equivalently, two equations in (5.4) define a 1-dimensional complex manifold containing $u = v = u' = v' = 0$. Thus the Jacobian of (5.4) at $u = u' = 0$, which is equal to

$$\begin{pmatrix} a_1 + b_1 \tau & c_1 + d_1 \tau' \\ a_2 + b_2 \tau & c_2 + d_2 \tau' \end{pmatrix},$$

has rank 1. By Lemma 4.1, the result follows. □

Using the above lemma, we now prove Theorem 1.5. For the first half of the proof, we follow along the same lines given in the proof of Lemma 8.1 in [3].

Theorem 5.2. *Let $\mathcal{M}$ be a 1-cusped hyperbolic 3-manifold whose cusp shape is non-quadratic. Let $t_{p/q}$ (where $|t_{p/q}| > 1$) be the holonomy of the core geodesic of (p,q)-Dehn filling. For p/q and p'/q' such that $|p|+|q|$ and $|p'|+|q'|$ are sufficiently large, if*

$$t_{p/q} = t_{p'/q'} \tag{5.5}$$

then

$$p/q = p'/q'. \tag{5.6}$$

Proof. Let

$$P = \big(t_{p/q}^{-q}, t_{p/q}^{p}, t_{p/q}^{-q'}, t_{p/q}^{p'}\big),$$

and then, by Theorem 3.11, there exists an universal constant B such that

$$h(P) \leq B.$$

Let

$$\mathbf{v} = (-q, p, -q', p'),$$

and then, by the properties of height (i.e. Theorem 3.2 and (3.1)), we can find c_1 such that

$$|\mathbf{v}| h(t_{p/q}) \leq c_1 B. \tag{5.7}$$

By Siegel's lemma, there exists $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3} \in \mathbb{Z}^4$ which vanishes at

$$-qx_1 + px_2 - q'x_3 + p'x_4 = 0 \tag{5.8}$$

and satisfies

$$\begin{aligned} |\mathbf{b_1}||\mathbf{b_2}||\mathbf{b_3}| &\leq |\mathbf{v}|, \\ |\mathbf{b_1}| \leq |\mathbf{b_2}| &\leq |\mathbf{b_3}|. \end{aligned} \tag{5.9}$$

Let $\mathbf{b_i} = (a_i, b_i, c_i, d_i)$ $(i = 1, 2)$ and H be defined by

$$\begin{aligned} M^{a_1} L^{b_1} (M')^{c_1} (L')^{d_1} &= 1, \\ M^{a_2} L^{b_2} (M')^{c_2} (L')^{d_2} &= 1. \end{aligned}$$

We first consider the case that P is an isolated point of $(\mathcal{X} \times \mathcal{X}) \cap H$.

Claim 5.3. *If P is an isolated point of $(\mathcal{X} \times \mathcal{X}) \cap H$, then the degree of P is bounded by some number which depends only on $\mathcal{X}$.*

Proof. By standard degree theory in arithmetic geometry, it is well-known that the degree of H is bound by $c_2|\mathbf{b_1}||\mathbf{b_2}|$ for some constant c_2, and thus by $c_2|\mathbf{v}|^{2/3}$ by (5.9). By Bézout's theorem, the degree D of P is bounded by the product of the degrees of $\mathcal{X} \times \mathcal{X}$ and H. Thus we have

$$D \leq c_3 |\mathbf{v}|^{2/3} \tag{5.10}$$

for some constant c_3 depending on $\mathcal{X}$. By Lemma 3.13,

$$h(t_{p/q}) \geq \frac{1}{c_4 D (\log 3D)^{\kappa}}$$

for some κ and c_4. Combining with (5.7), we deduce $|\mathbf{v}| \leq c_5 D(\log 3D)^{\kappa} B$ for some constant c_5 and, together with (5.10), we get $D \leq c_6\big(D(\log 3D)^{\kappa} B\big)^{2/3}$ for some constant c_6 depending only on $\mathcal{X}$. This completes the proof. $\square$

Next we consider the case the component of $(\mathcal{X} \times \mathcal{X}) \cap H$ containing P is an anomalous subvariety of $\mathcal{X} \times \mathcal{X}$. We denote this anomalous subvariety by $\mathcal{Y}$. If $\mathcal{Y}$ contains z^0, then, by Lemma 5.1, we get $p/q = p'/q'$ as desired. If $\mathcal{Y}$ does not contain z^0 and $\mathcal{X} \times \mathcal{X}$ has only a finite number of anomalous subvarieties near z^0, then, by shrinking the size of a neighborhood if necessary, we exclude those Dehn filling points contained in $\mathcal{Y}$.

Now we assume $\mathcal{X} \times \mathcal{X}$ contains infinitely many anomalous subvarieties near z^0 and each contains a Dehn filling point coming from two isometric Dehn filled manifolds of different filling coefficients. More precisely, we consider the following situation. Let $(p_i/q_i)_{i\in\mathcal{I}}$ and $(p'_i/q'_i)_{i\in\mathcal{I}}$ be two infinite sequences of co-prime pairs such that, for each i,

$$(p_i/q_i) \neq (p'_i/q'_i), \tag{5.11}$$

but

$$t_i = t'_i,$$

where t_i (resp. t'_i) is the holonomy of the core geodesic of $\mathcal{M}(p_i/q_i)$ (resp. $\mathcal{M}(p'_i/q'_i)$). Let

$$P_i = \Big(t_i^{-q_i}, t_i^{p_i}, (t'_i)^{-q'_i}, (t'_i)^{p'_i}\Big)$$

be the Dehn filling point in $\mathcal{X} \times \mathcal{X}$ (associated to $\mathcal{M}(p_i/q_i)$ and $\mathcal{M}(p'_i/q'_i)$), and H_i be an algebraic subgroup containing P_i and obtained by the same procedure given above (i.e. using Siegel's lemma). Let H_i be defined by the following equations

$$\begin{aligned} M^{a_{1i}} L^{b_{1i}} (M')^{a'_{1i}} (L')^{b'_{1i}} &= 1, \\ M^{a_{2i}} L^{b_{2i}} (M')^{a'_{2i}} (L)^{b'_{2i}} &= 1 \end{aligned} \tag{5.12}$$

for each $i \in \mathcal{I}$. We further assume that the component of $(\mathcal{X} \times \mathcal{X}) \cap H_i$ (say $\mathcal{Y}_i$) containing the Dehn filling point P_i is an anomalous subvariety of $\mathcal{X} \times \mathcal{X}$, and any neighborhood of z^0 contains infinitely many $\mathcal{Y}_i$. Then we have the following claim:

Claim 5.4. *For each i, the component of $(\mathcal{X} \times \mathcal{X}) \cap H_i$ containing z^0 is an anomalous subvariety of $\mathcal{X} \times \mathcal{X}$. (Thus each $(\mathcal{X} \times \mathcal{X}) \cap H_i$ contains at least two anomalous subvarieties.)*

Proof. Let b be a number such that there are infinitely many $\mathcal{Y}_i$ with each $\mathcal{Y}_i$ is a b-anomalous subvariety of $\mathcal{X} \times \mathcal{X}$ but not a $(b+1)$-anomalous subvariety. By Lemma 3.9, we find an algebraic subgroup $H^{(0)}$ such that, for each i, $\mathcal{Y}_i$ is contained in $(\mathcal{X} \times \mathcal{X}) \cap g_i H^{(0)}$ for some g_i. By Theorem 3.7 (i), the component of $(\mathcal{X} \times \mathcal{X}) \cap H^{(0)}$ containing z^0 (say $\mathcal{Y}^{(0)}$) is an anomalous subvariety of $\mathcal{X} \times \mathcal{X}$.

We claim $g_i H^{(0)} \subset H_i$ for each i. Suppose $g_i H^{(0)} \not\subset H_i$. Since $\mathcal{Y}_i \subset (\mathcal{X} \times \mathcal{X}) \cap H_i$ and $\mathcal{Y}_i \subset (\mathcal{X} \times \mathcal{X}) \cap g_i H^{(0)}$, we have $\mathcal{Y}_i \subset (\mathcal{X} \times \mathcal{X}) \cap (H_i \cap g_i H^{(0)})$. If $g_i H^{(0)} \not\subset H_i$, then $H_i \cap g_i H^{(0)}$ is an algebraic coset whose dimension is less than $g_i H^{(0)}$. But this contradicts the maximality of b. Thus $g_i H^{(0)} \subset H_i$ and this further implies $H^{(0)} \subset H_i$ for each i. So $(\mathcal{X} \times \mathcal{X}) \cap H^{(0)} \subset (\mathcal{X} \times \mathcal{X}) \cap H_i$ and $\mathcal{Y}^{(0)} \subset (\mathcal{X} \times \mathcal{X}) \cap H_i$. $\square$

Thus, again by Lemma 5.1, we have $p_i/q_i = p'_i/q'_i$ for each i, which contradicts the assumption (5.11).

In conclusion, if (5.5) holds, then either (5.6) holds or the degree of $t_{p/q}$ is bounded. By Theorem 3.11, the height of $t_{p/q}$ is uniformly bounded and, by Northcott's theorem, there are

only a finite number of choices for $t_{p/q}$. Combining with Theorem 2.5, we conclude there are only a finite number of Dehn filling coefficients having the same holonomy. Thus, except for those finitely many choices, the only case that makes (5.5) possible is (5.6). This completes the proof.

□

Remark. The above proof is the prototype of the other proofs below. Basically we will follow the same strategies given here to prove Theorem 1.7 as well as Theorem 1.9. In particular, Claims 5.3 and 5.4 will be used repeatedly there.

6. Proof of Theorem 1.7

Lemma 6.1. *Let M be a 2-cusped hyperbolic 3-manifold having rationally independent cusp shapes and $\mathcal{X}$ be its holonomy variety. Let H be a 2-dimensional algebraic subgroup defined by*

$$M_1^{a_1}L_1^{b_1}M_2^{c_1}L_2^{d_1} = 1,$$
$$M_1^{a_2}L_1^{b_2}M_2^{c_2}L_2^{d_2} = 1.$$

Suppose that (p_1, q_1) and (p_2, q_2) are two nonzero pairs (i.e. $p_i, q_i \neq 0$ where $i = 1, 2$) such that

$$\begin{aligned} -q_1a_1 + p_1b_1 - q_2c_1 + p_2d_1 = 0, \\ -q_1a_2 + p_1b_2 - q_2c_2 + p_2d_2 = 0. \end{aligned} \tag{6.1}$$

Then z^0 is an isolated component of $\mathcal{X} \cap H$.

Proof. Taking logarithm to each coordinate, H is locally isomorphic to the linear space defined by the following equations:

$$a_1u_1 + b_1v_1 + c_1u_2 + d_1v_2 = 0, \tag{6.2}$$
$$a_2u_1 + b_2v_1 + c_2u_2 + d_2v_2 = 0. \tag{6.3}$$

Since $\text{Def}(\mathcal{M}) \times \text{Def}(\mathcal{M})$ are parametrized by

$$v_1 = h_1(u_1, u_2) = \tau_1u_1 + m_1u_1^3 + \cdots, \quad v_2 = h_2(u_1, u_2) = \tau_2u_2 + m_2u_2^3 + \cdots,$$

$(\mathcal{X} \times \mathcal{X}) \cap H$ is locally (near z^0) biholomorphic to the complex manifold defined by

$$\begin{aligned} a_1u_1 + b_1(\tau_1u_1 + m_1u_1^3 + \cdots) + c_1u_2 + d_1(\tau_2u_2 + m_2u_2^3 + \cdots) = 0, \\ a_2u_1 + b_2(\tau_1u_1 + m_1u_1^3 + \cdots) + c_2u_2 + d_2(\tau_2u_2 + m_2u_2^3 + \cdots) = 0. \end{aligned} \tag{6.4}$$

If the component of $(\mathcal{X}\times\mathcal{X})\cap H$ containing z^0 is an anomalous subvariety, then, equivalently, two equations in (6.4) defines a 1-dimensional complex manifold containing $u_1 = v_1 = u_2 = v_2 = 0$. Thus the rank of the Jacobian matrix of (6.4) at $u_1 = u_2 = 0$, which is equal to

$$\begin{pmatrix} a_1 + b_1\tau_1 & c_1 + d_1\tau_2 \\ a_2 + b_2\tau_1 & c_2 + d_2\tau_2 \end{pmatrix},$$

is 1 or 0. However, this is impossible by Lemma 4.2. □

Theorem 6.2. *Let $\mathcal{M}$ a hyperbolic 2-cusped manifold and $\mathcal{X}$ be its holonomy variety. Let $t^1_{(p_1/q_1,p_2/q_2)}$ and $t^2_{(p_1/q_1,p_2/q_2)}$ be two holonomies of $(p_1/q_1, p_2/q_2)$-Dehn filling. If $|p_i|+|q_i|$ ($i = 1, 2$) are sufficietnly large, then $t^1_{(p_1/q_1,p_2/q_2)}$ and $t^2_{(p_1/q_1,p_2/q_2)}$ are multiplicatively independent.*

Proof. To simplify the notation we denote $t^1_{(p_1/q_1,p_2/q_2)}$ and $t^2_{(p_1/q_1,p_2/q_2)}$ by t_1 and t_2 respectively. If they are multiplicatively dependent, then

$$t_1 = \xi^{s_1}\eta^{e_1}, \quad t_2 = \xi^{s_2}\eta^{e_2} \tag{6.5}$$

for some η such that $|\eta| \neq 1$, a N-th primitive root of unity ξ and $s_i, e_i \in \mathbb{Z}$ $(1 \leq i \leq 2)$. We denote the corresponding Dehn filling point P by

$$\begin{aligned} P = \left(t_1^{-q_1}, t_1^{p_1}, t_2^{-q_2}, t_2^{p_2}\right) &= \left((\xi^{s_1}\eta^{e_1})^{-q_1}, (\xi^{s_1}\eta^{e_1})^{p_1}, (\xi^{s_2}\eta^{e_2})^{-q_2}, (\xi^{s_2}\eta^{e_2})^{p_2}\right) \\ &= \left(\xi^{l_1}\eta^{-e_1q_1}, \xi^{l_2}\eta^{e_1p_1}, \xi^{l_3}\eta^{-e_2q_2}, \xi^{l_4}\eta^{e_2p_2}\right) \end{aligned}$$

where $0 \leq l_i < N$ $(1 \leq i \leq 4)$, and consider the following form:

$$NX_0 + l_1X_1 + l_1X_2 + l_2X_3 + l_2X_4, \quad -e_1q_1X_1 + e_1p_1X_2 - e_2q_2X_3 + e_2p_2X_4$$

in the 5 variables X_0, X_1, X_2, X_3, X_4. By Siegel's lemma, we can make these forms vanish at independent points $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ in $\mathbb{Z}^5$ whose lengths satisfy $|\mathbf{b_1}||\mathbf{b_2}||\mathbf{b_3}| \leq c_2N|\mathbf{v}|$ (for some c_2) and $|\mathbf{b_1}| \leq |\mathbf{b_2}| \leq |\mathbf{b_3}|$ where $\mathbf{v} = (-e_1q_1, e_1p_1, -e_2q_2, e_2p_2)$. Writing

$$\begin{aligned} \mathbf{b_1} &= (b_{01}, b_{11}, b_{21}, b_{31}, b_{41}), \\ \mathbf{b_2} &= (b_{01}, b_{12}, b_{22}, b_{32}, b_{42}), \end{aligned}$$

we get the relations

$$\begin{aligned} \left(\xi^{l_1}\eta^{-e_1q_1}\right)^{b_{11}}\left(\xi^{l_1}\eta^{e_1p_1}\right)^{b_{21}}\left(\xi^{l_2}\eta^{-e_2q_2}\right)^{b_{31}}\left(\xi^{l_2}\eta^{-e_2q_2}\right)^{b_{41}} &= 1, \\ \left(\xi^{l_1}\eta^{-e_1q_1}\right)^{b_{12}}\left(\xi^{l_1}\eta^{e_1p_1}\right)^{b_{22}}\left(\xi^{l_2}\eta^{-e_2q_2}\right)^{b_{32}}\left(\xi^{l_2}\eta^{-e_2q_2}\right)^{b_{42}} &= 1. \end{aligned}$$

Thus the Dehn filling point P lies in an algebraic torus H defined by

$$\begin{aligned} M_1^{b_{11}}L_1^{b_{21}}M_2^{b_{31}}L_2^{b_{41}} &= 1, \\ M_1^{b_{12}}L_1^{b_{22}}M_2^{b_{32}}L_2^{b_{42}} &= 1. \end{aligned}$$

If P is an isolated point of $\mathcal{X} \cap H$, then following the same procedure given in the proof of Claim 5.3, we get that the degree of P is bounded by some constant depending only on $\mathcal{X}$.

Now we assume the component of $\mathcal{X} \cap H$ containing P is an anomalous subvariety of $\mathcal{X}$, and denote this anomalous subvariety by $\mathcal{Y}$. First note that, in this case, $\mathcal{Y}$ does not contain z^0 by Lemma 6.1. So if $\mathcal{X}$ has only a finite number of anomalous subvarieties near z^0 and $\mathcal{Y}$ is one of them, then, by shrinking the size of a neighborhood of z^0 if necessary, we discard all the Dehn filing points contained in $\mathcal{Y}$.

Next we consider the case $\mathcal{X}$ contains infinitely many anomalous subvarieties near z^0 and each contains a Dehn filling point coming from multiplicatively dependent holonomies. Similar to the proof Theorem 1.5, we suppose the following situation. Let $(p_{1i}, q_{1i})_{i\in\mathcal{I}}$ and $(p_{2i}, q_{2i})_{i\in\mathcal{I}}$ be there are two infinite sequences of co-prime pairs such that, for each $i \in \mathcal{I}$,

$$p_{1i}/q_{1i} \neq p_{2i}/q_{2i} \tag{6.6}$$

but two holonomies of $\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})$ are multiplicatively dependent. Let H_i be a 2-dimensional algebraic subgroup containing a Dehn filling point of $\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/_{2i})$ and constructed via the same procedure above (i.e. using Siegel's lemma). We denote the component of $\mathcal{X} \cap H_i$ containing P_i by $\mathcal{Y}_i$, and assume $\{\mathcal{Y}_i\}_{i\in\mathcal{I}}$ is a family of infinitely many anomalous subvarieties near z^0. Then, by Claim 5.4, the component of $\mathcal{X} \cap H_i$ containing z^0 is an anomalous subvariety of $\mathcal{X}$ for each i. However this is impossible unless $p_{1i}/q_{1i} = p_{2i}/q_{2i}$ (which contradicts (6.6)) by Lemma 6.1.

Similar to the conclusion of the proof of Theorem 1.5, we get the desired result by removing those finitely many Dehn filling points of bounded degree. □

Now we prove Theorem 1.7 using the above theorem.

Theorem 6.3. *Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold having non-quadratic, rationally independent cusp shapes and whose two cusps are SGI. Let*

$$\left\{t^1_{(p_1/q_1,p_2/q_2)}, t^2_{(p_1/q_1,p_2/q_2)}\right\}$$

be the set of holonomies of $(p_1/q_1, p_2/q_2)$-Dehn filling. If

$$\left\{t^1_{(p_1/q_1,p_2/q_2)}, t^2_{(p_1/q_1,p_2/q_2)}\right\} = \left\{t^1_{(p'_1/q'_1,p'_2/q'_2)}, t^2_{(p'_1/q'_1,p'_2/q'_2)}\right\}$$

for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ $(1 \leq i \leq 2)$, then

$$\left(p_1/q_1, p_2/q_2\right) = \left(p'_1/q'_1, p'_2/q'_2\right).$$

Proof. First, we view $\mathcal{X}$ as $\mathcal{X}_1 \times \mathcal{X}_2$ where $\mathcal{X}_i$ is the anomalous subvariety of $\mathcal{X}$ contained in $M_i = L_i = 1$ $(1 \leq i \leq 2)$. Geometrically, $\mathcal{X}_i$ represents the variety obtained by keeping i-th cusp complete. (So each $\mathcal{X}_i$ is considered as the holonomy variety of a 1-cusped manifold.)

To simplify the notation, we denote $\left(t^1_{(p_1/q_1,p_2/q_2)}, t^2_{(p_1/q_1,p_2/q_2)}\right)$ and $\left(t^1_{(p'_1/q'_1,p'_2/q'_2)}, t^2_{(p'_1/q'_1,p'_2/q'_2)}\right)$ by (t_1, t_2) and (t'_1, t'_2) respectively. By Theorem 1.5, if $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ $(i = 1, 2)$ are sufficiently large, then

$$t_1 \neq t'_1, \quad t_2 \neq t'_2.$$

On the other hand, if

$$t_1 = t'_2 \quad (\text{resp. } t'_1 = t_2),$$

it implies two holonomies of $(p_1/q_1, p'_2/q'_2)$-Dehn filled manifold (resp. $(p'_1/q'_1, p_2/q_2)$-Dehn filled manifold) are equal, which contradicts Theorem 6.2. Thus,

$$t_1 \neq t'_2, \quad t_2 \neq t'_1$$

for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$. This completes the proof. □

7. Proof of Theorem 1.9

Lemma 7.1. *Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold having non-quadratic, rationally independent cusp shapes and $\mathcal{X}$ be its holonomy variety. Let H be an algebraic subgroup defined by the following forms of equations*

$$\begin{aligned}
M_1^{a_1} L_1^{b_1} (M'_1)^{a'_1} (L'_1)^{b'_1} &= 1,\\
M_1^{a_2} L_1^{b_2} (M'_1)^{a'_2} (L'_1)^{b'_2} &= 1,\\
M_2^{c_1} L_2^{d_1} (M'_2)^{c'_1} (L'_2)^{d'_1} &= 1,\\
M_2^{c_2} L_2^{d_2} (M'_2)^{c'_2} (L'_2)^{d'_2} &= 1,
\end{aligned}$$

and $(p_1, q_1), (p_2, q_2), (p'_1, q'_1), (p'_2, q'_2)$ be four co-prime pairs satisfying

$$\begin{aligned}
-q_1a_1 + p_1b_1 - q'_1a'_1 + p'_1b'_1 &= 0,\\
-q_1a_2 + p_1b_2 - q'_1a'_2 + p'_1b'_2 &= 0,\\
-q_2c_1 + p_2d_1 - q'_2c'_1 + p'_2d'_1 &= 0,\\
-q_2c_2 + p_2d_2 - q'_2c'_2 + p'_2d'_2 &= 0.
\end{aligned}$$

Let $\mathcal{Y}$ be the component of $(\mathcal{X}\times\mathcal{X})\cap H$ containing z^0, and suppose it is an anomalous subvariety of $\mathcal{X}\times\mathcal{X}$.

If $\mathcal{Y}$ is a 1-dimensional anomalous subvariety, then $\mathcal{Y}$ is contained in either

$$M_1=M_1',L_1=L_1',\quad M_2=M_2'=L_2=L_2'=1$$

or

$$M_1=(M_1')^{-1},L_1=(L_1')^{-1},\quad M_2=M_2'=L_2=L_2'=1$$

or

$$M_2=M_2',L_2=L_2',\quad M_1=M_1'=L_1=L_1'=1$$

or

$$M_2=(M_2')^{-1},L_2=(L_2')^{-1},\quad M_1=M_1'=L_1=L_1'=1.$$

Moreover, in each case, we have

$$(p_1,q_1)=(p_1',q_1'),\quad a_i=-a_i',b_i=-b_i'\ (i=1,2)$$

or

$$(p_1,q_1)=-(p_1',q_1'),\quad a_i=a_i',b_i=b_i'\ (i=1,2)$$

or

$$(p_2,q_2)=(p_2',q_2'),\quad c_i=-c_i',d_i=-d_i'\ (i=1,2)$$

or

$$(p_2,q_2)=-(p_2',q_2'),\quad c_i=c_i',d_i=d_i'\ (i=1,2)$$

respectively.

If $\mathcal{Y}$ is a 2-dimensional anomalous subvariety, then we have either

$$(p_1,q_1)=(p_1',q_1'),\quad (p_2,q_2)=(p_2',q_2')$$

or

$$(p_1,q_1)=(p_1',q_1'),\quad (p_2,q_2)=-(p_2',q_2')$$

or

$$(p_1,q_1)=-(p_1',q_1'),\quad (p_2,q_2)=(p_2',q_2')$$

or

$$(p_1,q_1)=-(p_1',q_1'),\quad (p_2/q_2)=-(p_2'/q_2').$$

Proof. By taking logarithm to each coordinate, H is biholomorphic to the linear space defined by

$$\begin{aligned}
a_1u_1+b_1v_1+a_1'u_1'+b_1'v_1'&=0,\\
a_2u_1+b_2v_1+a_2'u_1'+b_2'v_1'&=0,\\
c_1u_2+d_1v_2+c_1'u_2'+d_1'v_2'&=0,\\
c_2u_2+d_2v_2+c_2'u_2'+d_2'v_2'&=0.
\end{aligned}$$

Let

$$\begin{aligned}
v_1 &= h_1(u_1,u_2) = \tau_1 u_1 + m_1 u_1^3 + m_3 u_1 u_2^2 + \cdots,\\
v_2 &= h_2(u_1,u_2) = \tau_2 u_2 + m_2 u_2^3 + m_3 u_1^2 u_2 + \cdots,\\
v_1' &= h_1(u_1',u_2') = \tau_1 u_1' + m_1 (u_1')^3 + m_3 u_1' (u_2')^2 + \cdots,\\
v_2' &= h_2(u_1',u_2') = \tau_2 u_2' + m_2 (u_2')^3 + m_3 (u_1')^2 u_2' + \cdots.
\end{aligned}$$

Then, near z^0, $\mathcal{Y}$ is locally biholomorphic to the complex manifold defined by

$$\begin{aligned}
&a_1u_1 + b_1\big(\tau_1 u_1 + m_1 u_1^3 + m_3 u_1 u_2^2 + \cdots\big) + a_1'u_1' + b_1'\big(\tau_1 u_1' + m_1(u_1')^3 + m_3 u_1'(u_2')^2 + \cdots\big) = 0,\\
&a_2u_1 + b_2\big(\tau_1 u_1 + m_1 u_1^3 + m_3 u_1 u_2^2 + \cdots\big) + a_2'u_1' + b_2'\big(\tau_1 u_1' + m_1(u_1')^3 + m_3 u_1'(u_2')^2 + \cdots\big) = 0,\\
&c_1u_2 + d_1\big(\tau_2 u_2 + m_2 u_2^3 + m_3 u_1^2 u_2 + \cdots\big) + c_1'u_2' + d_1'\big(\tau_2 u_2' + m_2(u_2')^3 + m_3 (u_1')^2 u_2' + \cdots\big) = 0,\\
&c_2u_2 + d_2\big(\tau_2 u_2 + m_2 u_2^3 + m_3 u_1^2 u_2 + \cdots\big) + c_2'u_2' + d_2'\big(\tau_2 u_2' + m_2(u_2')^3 + m_3 (u_1')^2 u_2' + \cdots\big) = 0.
\end{aligned} \tag{7.1}$$

Since $\mathcal{Y}$ is an anomalous subvariety containing z^0, (7.1) define a complex manifold of non-trivial dimension containing $(0,0,0,0)$. More precisely, if $\mathcal{Y}$ is 1-dimensional variety, then, by the implicit function theorem, the rank of the Jacobian of (7.1) at $(u_1,u_1',u_2,u_2')=(0,0,0,0)$, which is

$$\begin{pmatrix}
a_1+b_1\tau_1 & a_1'+b_1'\tau_1 & 0 & 0\\
a_2+b_2\tau_1 & a_2'+b_2'\tau_1 & 0 & 0\\
0 & 0 & c_1+d_1\tau_2 & c_1'+d_1'\tau_2\\
0 & 0 & c_2+d_2\tau_2 & c_2'+d_2'\tau_2
\end{pmatrix}, \tag{7.2}$$

is equal to 3. Likewise, if $\mathcal{Y}$ is a 2-dimensional variety, the rank of (7.2) is equal to 2.

First if the rank of (7.2) is 3, then the rank of one of the following two matrices is 1 and the other is 2:

$$\begin{pmatrix}
a_1+b_1\tau_1 & a_1'+b_1'\tau_1\\
a_2+b_2\tau_1 & a_2'+b_2'\tau_1
\end{pmatrix},$$

$$\begin{pmatrix}
c_1+d_1\tau_2 & c_1'+d_1'\tau_2\\
c_2+d_2\tau_2 & c_2'+d_2'\tau_2
\end{pmatrix}.$$

Without loss of generosity, we assume the rank of the first one is 1 and the rank of the second one is 2. Then, by Lemma 4.1, we have

$$\begin{gathered}
(p_1,q_1) = (p_1',q_1'),\\
a_1=-a_1', b_1=-b_1', a_2=-a_2', b_2=-b_2',
\end{gathered} \tag{7.3}$$

or

$$\begin{gathered}
(p_1,q_1) = -(p_1',q_1'),\\
a_1=a_1', b_1=b_1', a_2=a_2', b_2=b_2'.
\end{gathered} \tag{7.4}$$

Since the rank of the following matrix

$$\begin{pmatrix}
c_1+d_1\tau_2 & c_1'+d_1'\tau_2\\
c_2+d_2\tau_2 & c_2'+d_2'\tau_2
\end{pmatrix}$$

is equal to 2, by the implicit function theorem, we get $u_2=0, u_2'=0$ (i.e. $M_2=M_2'=1$). Now it is clear

$$u_1 = u_1', \quad u_2 = u_2' = 0$$

or

$$u_1 = -u_1', \quad u_2 = u_2' = 0$$

parametrizes the complex manifold defined in (7.2) under (7.3) or (7.4) respectively. Equivalently, this implies $\mathcal{Y}$ is contained in either

$$M_1 = M_1', \quad M_2 = M_2' = L_2 = L_2' = 1$$

or

$$M_1 = (M_1')^{-1}, \quad M_2 = M_2' = L_2 = L_2' = 1.$$

Second, if the rank of (7.2) is equal to 2, then the rank of the following two matrices is equal to 1:

$$\begin{pmatrix} a_1 + b_1\tau_1 & a_1' + b_1'\tau_1 \\ a_2 + b_2\tau_1 & a_2' + b_2'\tau_1 \end{pmatrix},$$

$$\begin{pmatrix} c_1 + d_1\tau_2 & c_1' + d_1'\tau_2 \\ c_2 + d_2\tau_2 & c_2' + d_2'\tau_2 \end{pmatrix}.$$

Again, by Lemma 4.1, the result follows. □

Lemma 7.2. *Let $\mathcal{X}$ be the same as the one in the previous lemma. Let H be an algebraic subgroup defined by*

$$\begin{aligned} M_1^{a_1}L_1^{b_1}(M_2')^{a_1'}(L_2')^{b_1'} &= 1, \\ M_1^{a_2}L_1^{b_2}(M_2')^{a_1'}(L_2')^{b_2'} &= 1, \\ M_2^{c_1}L_2^{d_1}(M_1')^{c_1'}(L_1')^{d_1'} &= 1, \\ M_2^{c_2}L_2^{d_2}(M_1')^{c_2'}(L_1')^{d_2'} &= 1 \end{aligned}$$

and $(p_1, q_1), (p_2, q_2), (p_1', q_1'), (p_2', q_2')$ be four co-prime pairs satisfying

$$\begin{aligned} -q_1a_1 + p_1b_1 - q_1'a_1' + p_1'b_1' &= 0, \\ -q_1a_2 + p_1b_2 - q_1'a_2' + p_1'b_2' &= 0, \\ -q_2c_1 + p_2d_1 - q_2'c_1' + p_2'd_1' &= 0, \\ -q_2c_2 + p_2d_2 - q_2'c_2' + p_2'd_2' &= 0. \end{aligned}$$

Then z^0 is an isolated component of $(\mathcal{X} \times \mathcal{X}) \cap H$.

Proof. Let $\mathcal{Y}$ be the component of $(\mathcal{X} \times \mathcal{X}) \cap H$ containing z^0. By taking logarithm to each coordinate, H is biholomorphic to the linear space defined by

$$\begin{aligned} a_1u_1 + b_1v_1 + a_1'u_2' + b_1'v_2' &= 0, \\ a_2u_1 + b_2v_1 + a_2'u_2' + b_2'v_2' &= 0, \\ c_1u_2 + d_1v_2 + c_1'u_1' + d_1'v_1' &= 0, \\ c_2u_2 + d_2v_2 + c_2'u_1' + d_2'v_1' &= 0. \end{aligned}$$

Let

$$v_1 = h_1(u_1, u_2) = \tau_1u_1 + m_1u_1^3 + \cdots, \quad v_2 = h_2(u_1, u_2) = \tau_2u_2 + m_2u_2^3 + \cdots,$$
$$v_1' = h_1(u_1', u_2') = \tau_1u_1' + m_1(u_1')^3 + \cdots, \quad v_2' = h_2(u_1', u_2') = \tau_2u_2' + m_2(u_2')^3 + \cdots.$$

Then, near z^0, $\mathcal{Y}$ is locally biholomorphic to the complex manifold defined by

$$\begin{aligned}
&a_1u_1 + b_1(\tau_1 u_1 + m_1 u_1^3 + \cdots) + a_1' u_2' + b_1'(\tau_2 u_2' + m_2(u_2')^3 + \cdots) = 0,\\
&a_2u_1 + b_2(\tau_1 u_1 + m_1 u_1^3 + \cdots) + a_2' u_2' + b_2'(\tau_2 u_2' + m_2(u_2')^3 + \cdots) = 0,\\
&c_1u_2 + d_1(\tau_2 u_2 + m_2 u_2^3 + \cdots) + c_1' u_1' + d_1'(\tau_1 u_1' + m_1(u_1')^3 + \cdots) = 0,\\
&c_2u_2 + d_2(\tau_2 u_2 + m_2 u_2^3 + \cdots) + c_2' u_1' + d_2'(\tau_1 u_1' + m_1(u_1')^3 + \cdots) = 0.
\end{aligned} \tag{7.5}$$

If $\mathcal{Y}$ is an anomalous subvariety containing z^0, (7.1) define a complex manifold of non-trivial dimension containing $(0,0,0,0)$. So the rank of the Jacobian of (7.5) at $(u_1, u_1', u_2, u_2') = (0,0,0,0)$, which is equal to

$$\begin{pmatrix} a_1 + b_1\tau_1 & a_1' + b_1'\tau_2 & 0 & 0 \\ a_2 + b_2\tau_1 & a_2' + b_2'\tau_2 & 0 & 0 \\ 0 & 0 & c_1 + d_1\tau_2 & c_1' + d_1'\tau_1 \\ 0 & 0 & c_2 + d_2\tau_2 & c_2' + d_2'\tau_1 \end{pmatrix}, \tag{7.6}$$

is less than 4. However, this is impossible by Lemma 4.2. $\square$

Now we prove Theorem 1.9.

Theorem 7.3. *Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold having non-quadratic and rationally independent cusp shapes. Let*

$$\Phi(u_1, u_2) = \tau_1 u_1^2 + \tau_2 u_2^2 + m_{40} u_1^4 + m_{22} u_1^2 u_2^2 + m_{04} u_2^4 + \cdots$$

be its Neumann-Zagier potential function such that $m_{22} \neq 0$. Let

$$\left(t^1_{(p_1/q_1, p_2/q_2)}, t^2_{(p_1/q_1, p_2/q_2)}\right)$$

be the set of holonomies of $(p_1/q_1, p_2/q_2)$-Dehn filling. If

$$\left\{t^1_{(p_1/q_1, p_2/q_2)}, t^2_{(p_1/q_1, p_2/q_2)}\right\} = \left\{t^1_{(p_1'/q_1', p_2'/q_2')}, t^2_{(p_1'/q_1', p_2'/q_2')}\right\}, \tag{7.7}$$

then

$$(p_1/q_1, p_2/q_2) = (p_1'/q_1', p_2'/q_2'). \tag{7.8}$$

Proof. To simplify the notation, we denote $\left(t^1_{(p_1/q_1, p_2/q_2)}, t^2_{(p_1/q_1, p_2/q_2)}\right)$ and $\left(t^1_{(p_1'/q_1', p_2'/q_2')}, t^2_{(p_1'/q_1', p_2'/q_2')}\right)$ by (t_1, t_2) and (t_1', t_2') respectively. The condition (7.7) implies either

$$t_1 = t_1', \quad t_2 = t_2' \tag{7.9}$$

or

$$t_1 = t_2', \quad t_2 = t_1'. \tag{7.10}$$

We show that, in both cases, either it satisfies (7.8) or the degree of (t_1, t_2) is uniformly bounded. Since the proof is fairly long, we consider each case separately. We treat (7.9) in **Part I** first and (7.10) in **Part II** later.

Part I Note that, by Lemma 6.2, t_1 and t_2 are multiplicatively independent. Let

$$\begin{aligned}
P_1 &= (M_1, L_1, M_1', L_1') = (t_1^{-q_1}, t_1^{p_1}, t_1^{-q_1'}, t_1^{p_1'}),\\
P_2 &= (M_2, L_2, M_2', L_2') = (t_2^{-q_2}, t_2^{p_2}, t_2^{-q_2'}, t_2^{p_2'}).
\end{aligned}$$

Using Siegel's lemma, we get two algebraic subgroups H_i $(i=1,2)$ defined by

$$\begin{aligned} M_i^{a_{i1}}L_i^{b_{i1}}(M_i')^{c_{i1}}(L_i')^{d_{i1}} &= 1,\\ M_i^{a_{i2}}L_i^{b_{i2}}(M_i')^{c_{i2}}(L_i')^{d_{i2}} &= 1, \end{aligned} \tag{7.11}$$

such that P_i lies in an algebraic group defined by (7.11) and

$$|\mathbf{b_{i1}}||\mathbf{b_{i2}}| \leq |\mathbf{v_i}|^{2/3}$$

where

$$\begin{aligned} \mathbf{b_{ij}} &= (a_{ij}, b_{ij}, c_{ij}, d_{ij}),\\ \mathbf{v_i} &= (-q_i, p_i, -q_i', p_i') \end{aligned}$$

$(1 \leq i, j \leq 2)$. Let

$$H = H_1 \cap H_2,$$

and

$$P = \big(M_1, L_1, M_2, L_2, M_1', L_1', M_2', L_2'\big) = \big(t_1^{-q_1}, t_1^{p_1}, t_2^{-q_2}, t_2^{p_2}, t_1^{-q_1'}, t_1^{p_1'}, t_2^{-q_2'}, t_2^{p_2'}\big).$$

We first consider the case P is an isolated point of $(\mathcal{X}\times\mathcal{X})\cap H$. Similar to Claim 5.3, in this case, the degree of P is uniformly bounded as the following claim shows:

Claim 7.4. *If P is an isolated point of $(\mathcal{X}\times\mathcal{X})\cap H$, then the degree of P is bounded by D depending only on $\mathcal{X}$.*

Proof. By Theorem 3.11, there exists an universal constant B such that $h(P)\leq B$ and, by the properties of height, we can find c_1 such that

$$|\mathbf{v_1}|h(t_1) \leq c_1 B, \tag{7.12}$$

$$|\mathbf{v_2}|h(t_2) \leq c_1 B. \tag{7.13}$$

By standard degree theory, the degree of H is bounded by $c_2\prod_{i=1}^{2}|\mathbf{b_{i1}}||\mathbf{b_{i2}}|$, and so by $c_2(|\mathbf{v_1}||\mathbf{v_2}|)^{2/3}$ for some constant c_2. By Bézout's theorem, the degree D of the Dehn filling point P is bounded by the product of the degrees of $\mathcal{X}\times\mathcal{X}$ and H: that is

$$D \leq c_3\big(|\mathbf{v_1}||\mathbf{v_2}|\big)^{2/3} \tag{7.14}$$

where c_3 is a constant depending on $\mathcal{X}$. By Lemma 3.13,

$$h(t_1)h(t_2) \geq \frac{1}{c_4 D(\log 3D)^{\kappa}}$$

for some κ and c_4. On the other hand, by (7.12), we deduce $|\mathbf{v_1}||\mathbf{v_2}| \leq c_5 D(\log 3D)^{\kappa}B^2$ for some constant c_5, and thus, combining with (7.14), we get $D \leq c_6\big(D(\log 3D)^{\kappa}B^2\big)^{2/3}$ where c_6 is a constant depending only on $\mathcal{X}$. This completes the proof. □

Next we consider the case that the component of $(\mathcal{X}\times\mathcal{X})\cap H$ containing P is an anomalous subvariety of $\mathcal{X}\times\mathcal{X}$. We denote this component by $\mathcal{Y}$. If $\mathcal{Y}$ contains z^0, by Lemma 7.1, we get

$$(p_1/q_1, p_2/q_2) = (p_1'/q_1', p_2'/q_2').$$

or $\mathcal{Y}$ is contained in either

$$\begin{aligned} M_1 = (M_1')^{\pm1}, &\quad L_1 = (L_1')^{\pm1},\\ M_2 = M_2' &= L_2 = L_2' = 1, \end{aligned}$$

or

$$\begin{aligned} M_1 = M' = L_1 = L'_1 = 1, \\ M_2 = (M'_2)^{\pm 1}, \quad L_2 = (L'_2)^{\pm 1}. \end{aligned}$$

But the last two cases contradict to the fact that $t_1, t_2 \neq 1$. If $\mathcal{Y}$ does not contain z^0 and $\mathcal{X} \times \mathcal{X}$ contains only a finite number of anomalous subvarieties near z^0, then, by shrinking the size of a neighborhood of z^0, we exclude the Dehn filling points contained in $\mathcal{Y}$.

Now we assume $\mathcal{X} \times \mathcal{X}$ contains infinitely many anomalous subvarieties near z^0. Similar to the proofs of the two previous main theorems, we consider the situation as follows. Suppose $(p_{1i}/q_{1i}, p_{2i}/q_{2i})_{i\in\mathcal{I}}$ and $(p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})_{i\in\mathcal{I}}$ are two infinite sequences such that, for each $i \in \mathcal{I}$,

$$(p_{1i}/q_{1i}, p_{2i}/q_{2i}) \neq (p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i}), \tag{7.15}$$

and

$$t_{1i} = t'_{1i}, \quad t_{2i} = t'_{2i}$$

where $\{t_{1i}, t_{2i}\}$ and $\{t'_{1i}, t'_{2i}\}$ are the sets of holonomies of $\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})$ and $\mathcal{M}(p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})$ respectively. Let

$$P_i = \left(t_{1i}^{-q_{1i}}, t_{1i}^{p_{1i}}, t_{2i}^{-q_{2i}}, t_{2i}^{p_{2i}}, (t'_{1i})^{-q'_{1i}}, (t'_{1i})^{p'_{1i}}, (t'_{2i})^{-q'_{2i}}, (t'_{2i})^{p'_{2i}}\right)$$

be the Dehn filling point associated to $\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})$ and $\mathcal{M}(p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})$ in $\mathcal{X} \times \mathcal{X}$, and H_i be an algebraic subgroup containing P_i, obtained by the same procedure shown earlier (i.e. using Siegel's lemma). Thus H_i is defined by equations of the following form:

$$\begin{aligned} M_1^{a_{1i}} L_1^{b_{1i}} (M'_1)^{a'_{1i}} (L'_1)^{b'_{1i}} &= 1, \\ M_1^{a_{2i}} L_1^{b_{2i}} (M'_1)^{a'_{2i}} (L'_1)^{b'_{2i}} &= 1, \\ M_2^{c_{1i}} L_2^{d_{1i}} (M'_2)^{c'_{1i}} (L'_2)^{d'_{1i}} &= 1, \\ M_2^{c_{2i}} L_2^{d_{2i}} (M'_2)^{c'_{2i}} (L'_2)^{d'_{2i}} &= 1. \end{aligned} \tag{7.16}$$

We further assume that, for each $i \in \mathcal{I}$, the component of $(\mathcal{X} \times \mathcal{X}) \cap H_i$ containing P_i, denoted by $\mathcal{Y}_i$, is an anomalous subvariety of $\mathcal{X} \times \mathcal{X}$ near z^0, and $\{\mathcal{Y}_i\}_{i\in\mathcal{I}}$ are all different. That is, for a given small neighborhood of z^0, we have a family of infinitely many anomalous subvarieties $\{\mathcal{Y}_i\}_{i\in\mathcal{I}}$ all intersecting the neighborhood. By Claim 5.4, we find $H^{(0)}$ such that, for each i,

- $H^{(0)} \subset H_i$;
- $\mathcal{Y}_i \subset (\mathcal{X} \times \mathcal{X}) \cap g_i H^{(0)}$ with some g_i;
- $(\mathcal{X} \times \mathcal{X}) \cap H^{(0)}$ contains an anomalous subvariety $\mathcal{Y}^{(0)}$ of $\mathcal{X} \times \mathcal{X}$ containing z^0.[3]

First, if $\mathcal{Y}^{(0)}$ is a 2-dimensional anomalous subvariety, by Lemma 7.1, we have

$$p_{1i}/q_{1i} = p'_{1i}/q'_{1i}, \quad p_{2i}/q_{2i} = p'_{2i}/q'_{2i}$$

for each $i \in \mathcal{I}$. But this contradicts the initial assumption (7.15).

Second, if $\mathcal{Y}^{(0)}$ is a 1-dimensional anomalous subvariety, then, again by Lemma 7.1, $\mathcal{Y}^{(0)}$ is contained in either

$$M_1 = M'_1, \quad L_1 = L'_1, \quad M_2 = M'_2 = L_2 = L'_2 = 1, \tag{7.17}$$

[3] So $\mathcal{Y}^{(0)}$ is a component of $(\mathcal{X} \times \mathcal{X}) \cap H_i$ as well for each i.

or

$$M_1 = (M_1')^{-1}, \quad L_1 = (L_1')^{-1}, \quad M_2 = M_2' = L_2 = L_2' = 1,$$

or

$$M_2 = M_2', \quad L_2 = L_2', \quad M_1 = M_1' = L_1 = L_1' = 1,$$

or

$$M_2 = (M_2')^{-1}, \quad L_2 = (L_2')^{-1}, \quad M_1 = M_1' = L_1 = L_1' = 1.$$

Without loss of generality, we assume $\mathcal{Y}^{(0)}$ is contained in the algebraic subgroup defined by (7.17). So, for each $i \in \mathcal{I}$, we have

$$(p_{1i}, q_{1i}) = (p'_{1i}, q'_{1i}),$$

and

$$a_{1i} = -a'_{1i}, \quad b_{1i} = -b'_{1i}, \quad a_{2i} = -a'_{2i}, \quad b_{2i} = -b'_{2i}$$

in (7.16). In this case, since each P_i lies in the algebraic subgroup defined by $M_1 = M_1', L_1 = L_1'$, we can further assume

$$a_{1i} = -a'_{1i} = 1, \quad b_{1i} = -b'_{1i} = 0, \quad a_{2i} = -a'_{2i} = 0, \quad b_{2i} = -b'_{2i} = 1,$$

and so H_i is defined by the following simpler forms of equations:

$$\begin{aligned} M_1 = M_1', \quad L_1 = L_1',\\ M_2^{c_{1i}} L_2^{d_{1i}} (M_2')^{c'_{1i}} (L_2')^{d'_{1i}} = 1,\\ M_2^{c_{2i}} L_2^{d_{2i}} (M_2')^{c'_{2i}} (L_2')^{d'_{2i}} = 1, \end{aligned} \tag{7.18}$$

for each $i \in \mathcal{I}$.

Recall that $H^{(0)}$ is contained in (7.18) for every $i \in \mathcal{I}$. Now we have the following cases for $H^{(0)}$:

(1) $H^{(0)}$ is a 4-dimensional algebraic torus. That is, the vector space spanned by the following two vectors

$$(c_{1i}, d_{1i}, c'_{1i}, d'_{1i}), \ (c_{2i}, d_{2i}, c'_{2i}, d'_{2i}) \tag{7.19}$$

are all equal for every $i \in \mathcal{I}$;

(2) $H^{(0)}$ is a 6-dimensional algebraic torus. In this case $H^{(0)}$ is defined by

$$\begin{aligned} M_1 = M_1', \quad L_1 = L_1',\\ M_2 = M_2' = L_2 = L_2' = 1; \end{aligned}$$

(3) $H^{(0)}$ is a 5-dimensional algebraic torus.

We analyze each case step by step and show that each contradicts one of our initial assumptions.

In the first case (1), it implies that all the Dehn filling points $\{P_i\}_{i\in\mathcal{I}}$ are contained in $H^{(0)}$. But, this contradicts our assumption that $P_i \in \mathcal{Y}_i\big(\subset (\mathcal{X}\times\mathcal{X})\cap g_i H^{(0)}\big)$ and all the Y_i are different.

In the second case (2), we let $H^{(1)}$ and $H^{(2)}$ be algebraic tori defined by

$$M_1 = M_1', \quad L_1 = L_1',$$

and

$$M_2 = M_2' = L_2 = L_2' = 1$$

respectively, and represent $H^{(0)}$ as $H^{(1)} \cap H^{(2)}$. Since $H_i \subset H^{(1)}$ for all i, we can suppose, for each i, there exists g_i such that $\mathcal{Y}_i$ is contained in

$$(\mathcal{X} \cap \mathcal{X}) \cap (H^{(1)} \cap g_i H^{(2)}). \tag{7.20}$$

By moving to Def($\mathcal{M}$) $\times$ Def($\mathcal{M}$), equivalently, this implies there are infinitely many

$$(\xi_1, \xi_2, \xi_3, \xi_4) \in (\mathbb{C})^4$$

such that the intersection between Def($\mathcal{M}$) $\times$ Def($\mathcal{M}$) and the manifold defined by

$$\begin{gathered} u_1 = u_1', \quad v_1 = v_1', \\ u_2 = \xi_1, \quad u_2' = \xi_2, \quad v_2 = \xi_3, \quad v_2' = \xi_4 \end{gathered} \tag{7.21}$$

is a 1-dimensional complex manifold. In other words, if we let

$$\begin{aligned} \frac{1}{2}\frac{\partial \Phi}{\partial u_1}(u_1, u_2) = v_1 = h_1(u_1, u_2) = \tau_1 u_1 + 2m_{40}u_1^3 + m_{22}u_1u_2^2 + \cdots, \\ \frac{1}{2}\frac{\partial \Phi}{\partial u_2}(u_1, u_2) = v_2 = h_2(u_1, u_2) = \tau_2 u_2 + 2m_{04}u_2^3 + m_{22}u_2u_1^2 + \cdots, \end{aligned} \tag{7.22}$$

then

$$h_1(u_1, \xi_1) = h_1(u_1, \xi_2), \tag{7.23}$$

$$\xi_3 = h_2(u_1, \xi_1) = \tau_2\xi_1 + 2m_{04}\xi_1^3 + m_{22}\xi_1u_1^2 + \cdots, \tag{7.24}$$

$$\xi_4 = h_2(u_1, \xi_2) = \tau_2\xi_2 + 2m_{04}\xi_2^3 + m_{22}\xi_2u_1^2 + \cdots, \tag{7.25}$$

which are equivalent to (7.21), define a 1-dimensional complex manifold parametrized by u_1. However this is impossible since we have only finitely many possibilities for u_1 in (7.24) (or (7.25)). In other words, (7.23) - (7.25) define a 1-dimensional complex manifold if and only if $h_1(u_1, u_2)$ is independent of u_2 and $h_2(u_1, u_2)$ is independent of u_1. But this contradicts our assumption that two cusps of $\mathcal{M}$ is not SGI.

Now we consider the last case (3). In this case, we let $H^{(0)} = H^{(1)} \cap H^{(2)}$ where $H^{(1)}$ and $H^{(2)}$ are algebraic tori defined by

$$M_1 = M_1, \quad L_1 = L_1,$$

and

$$\begin{aligned} M_2^{a_1}L_2^{b_1}(M_2')^{c_1}(L_2')^{d_1} = 1, \\ M_2^{a_2}L_2^{b_2}(M_2')^{c_2}(L_2')^{d_2} = 1, \\ M_2^{a_3}L_2^{b_3}(M_2')^{c_3}(L_2')^{d_3} = 1 \end{aligned} \tag{7.26}$$

respectively. Applying Gauss elimination if necessary, we assume $H^{(2)}$ is contained in an algebraic subgroup defined by the following equations:

$$\begin{aligned} M_2^{a_1}L_2^{b_1}(M_2')^{c_1}(L_2')^{d_1} = 1, \\ M_2^{a_2}L_2^{b_2}(M_2')^{c_2} = 1, \\ M_2^{a_3}L_2^{b_3} = 1. \end{aligned} \tag{7.27}$$

Changing the basis if necessary,[4] we further assume (7.27) is of the following simpler forms of equations:

$$\begin{aligned} M_2^{a_1} L_2^{b_1} (M_2')^{c_1} (L_2')^{d_1} &= 1, \\ L_2^{b_2} (M_2')^{c_2} &= 1, \\ M_2 &= 1. \end{aligned} \tag{7.28}$$

By abusing the notation, we still use $H^{(2)}$ to denote the algebraic subgroup defined by (7.28). By the assumption, for each i, there exists g_i such that $\mathcal{Y}_i$ is contained in

$$(\mathcal{X} \times \mathcal{X}) \cap (H^{(1)} \cap g_i H^{(2)}). \tag{7.29}$$

Taking logarithm to each coordinate, $H^{(2)}$ is biholomorphic to

$$\begin{aligned} a_1 u_2 + b_1 v_2 + c_1 u_2' + d_1 v_2' &= 0, \\ b_2 v_2 + c_2 u_2' &= 0, \\ u_2 &= 0, \end{aligned}$$

and so, by letting

$$v_1 = h_1(u_1, u_2), \quad v_2 = h_2(u_1, u_2),$$

the variety in (7.29) is locally biholomorphic (near z^0) to the complex manifold defined by

$$\begin{aligned} u_1 = u_1', \quad v_1 &= v_1', \\ a_1 u_2 + b_1 h_2(u_1, u_2) + c_1 u_2' + d_1 h_2(u_1, u_2') &= \epsilon_{1i}, \\ b_2 h_2(u_1, u_2) + c_2 u_2' &= \epsilon_{2i}, \\ u_2 &= \epsilon_{3i} \end{aligned}$$

for some $(\epsilon_{1i}, \epsilon_{2i}, \epsilon_{3i}) \in \mathbb{C}^3$.[5] By Theorem 3.7, since the union of all $\mathcal{Y}_i$ is contained in a Zariski closed set, we use ϵ_{3i} as a parameter and represent ϵ_{1i} and ϵ_{2i} as holomorphic functions of ϵ_{3i}. Thus we get equations of the following types which define a 2-dimensional complex manifold:

$$h_1(u_1, u_2) = h_1(u_1, u_2'), \tag{7.30}$$

$$a_1 u_2 + b_1 h_2(u_1, u_2) + c_1 u_2' + d_1 h_2(u_1, u_2') = \phi'(u_2), \tag{7.31}$$

$$b_2 h_2(u_1, u_2) + c_2 u_2' = \phi(u_2). \tag{7.32}$$

Note that, since[6]

$$u_2' = \frac{1}{c_2}\phi(u_2) - \frac{b_2}{c_2} h_2(u_1, u_2),$$

the above complex manifold is parametrized by u_1 and u_2.

---

[4]Keep the basis of the first cusp the same and change the basis of the second cusp by letting

$$m_2^* = m_2^{a_3} l_2^{b_3}, \quad l_2^* = m_2^r l_2^s$$

where $a_3 s - b_3 r = 1$. We apply this basis change to both factors in $\mathcal{X} \times \mathcal{X}$, and get a new holonomy variety $\mathcal{X}^* \times \mathcal{X}^*$. Note that the Neumann-Zagier potential function obtained from this new basis also satisfies the given condition on one of its coefficients (i.e. $m_{22} \neq 0$).

[5]Note that this 1-dimensional complex manifold is parametrized by u_1 and so M_1 is a nontrivial coordinate function on $\mathcal{Y}_i$.

[6]If $c_2 = 0$, then, for each fixed u_2, there are only a finite number of choices for u_1 in (7.32). Also for fixed u_1 and u_2, there are only a finite number of choices for u_2' in (7.30) and (7.31), which contradicts the fact that (7.30) - (7.32) define a 2-dimensional complex manifold. Thus we can always assume $c_2 \neq 0$.

Now we prove the following claim:

Claim 7.5. *If* (7.30)-(7.32) *define a* 2*-dimensional complex manifold, then they are equivalent to either*

$$h_1(u_1, u_2) = h_1(u_1, u_2'), \quad h_2(u_1, u_2) = h_2(u_1, u_2'), \quad u_2' = u_2$$

or

$$h_1(u_1, u_2) = h_1(u_1, u_2'), \quad h_2(u_1, u_2) = -h_2(u_1, u_2'), \quad u_2' = -u_2.$$

Proof. We first start with the following subclaim:

Subclaim 1. In (7.31), we can assume $\phi'(u_2) = 0$.

Proof. Let $\mathcal{Z}$ the algebraic surface containing all the $\mathcal{Y}_i$. Let $\mathcal{Z}_\xi$ be the intersection between $\mathcal{Z}$ and the algebraic coset defined by $M_1 = \xi$ where ξ is a number sufficiently close to 1, and Proj $\mathcal{Z}_\xi$ be the image of $\mathcal{Z}_\xi$ under the following natural projection:

$$\text{Proj} \,:\, (z_1, \dots, z_n, M_1, \dots, L_2) \times (z_1', \dots, z_n', M_1', \dots, L_2') \longrightarrow (M_2, L_2, M_2', L_2').$$

Recall that, for each i, $\mathcal{Y}_i$ is the component of $\mathcal{X} \cap H_i$ containing P_i where H_i is defined by

$$\begin{gathered} M_1 = M_1', \quad L_1 = L_1', \\ M_2^{c_{1i}} L_2^{d_{1i}} (M_2')^{c_{1i}'} (L_2')^{d_{1i}'} = 1, \\ M_2^{c_{2i}} L_2^{d_{2i}} (M_2')^{c_{2i}'} (L_2')^{d_{2i}'} = 1. \end{gathered}$$

Since $\mathcal{Z}_\xi$ is the intersection of $\mathcal{Z}$ with $M_1 = \xi$, and, by the initial assumption, all the $\mathcal{Y}_i$ are different and contained in $\mathcal{Z}$, we get that Proj $\mathcal{Z}_\xi$ contains infinitely many different points where each of them is contained in

$$\begin{gathered} M_2^{c_{1i}} L_2^{d_{1i}} (M_2')^{c_{1i}'} (L_2')^{d_{1i}'} = 1, \\ M_2^{c_{2i}} L_2^{d_{2i}} (M_2')^{c_{2i}'} (L_2')^{d_{2i}'} = 1 \end{gathered}$$

for some i. Thus, by Theorem 3.15, Proj $\mathcal{Z}_\xi$ is contained in an algebraic subgroup. In other words, for each complex number ξ, Proj $\mathcal{Z}_\xi$ is contained in some algebraic subgroup H_ξ of co-dimension 1. Since there are uncountably many complex numbers but only countably many algebraic subgroups, we have infinitely many (indeed uncountably many) ξ such that Proj $\mathcal{Z}_\xi$ is contained in the same algebraic subgroup. Let H be an algebraic group containing infinitely many $\mathcal{Z}_\xi$. As H contains a Zariski-dense subset of $\mathcal{Z}$, it contains $\mathcal{Z}$ as well. Without loss of generality, we suppose H is defined by

$$M_2^{a_1} L_2^{b_1} (M_2')^{c_1} (L_2')^{d_1} = 1, \tag{7.33}$$

which is the first equation in (7.28). Since $\mathcal{Y}_i \subset H$ for all i, we indeed have $\phi'(u_2) = 0$ in (7.31). This completes the proof of the subclaim. $\square$

Now we rewrite (7.30) - (7.32) as follows:

$$h_1(u_1, u_2) - h_1(u_1, u_2') = 0, \tag{7.34}$$

$$a_1 u_2 + b_1 h_2(u_1, u_2) + c_1 u_2' + d_1 h_2(u_1, u_2') = 0, \tag{7.35}$$

$$b_2 h_2(u_1, u_2) + c_2 u_2' - \phi(u_2) = 0. \tag{7.36}$$

Since the above equations define 2-dimensional complex manifold, (7.34) - (7.36) are all equivalent. By (7.36), we represent u_2' as

$$u_2' = \frac{1}{c_2}\phi'(u_2) - \frac{b_2}{c_2}h(u_1, u_2). \tag{7.37}$$

To simplify the notation, we denote $\frac{1}{c_2}$ and $-\frac{b_2}{c_2}$ by c and b respectively. Plugging (7.37) into (7.34) and (7.35), we get

$$h_1(u_1, u_2) - h_1\big(u_1, c\phi'(u_2) + bh_2(u_1, u_2)\big) = 0, \tag{7.38}$$

$$a_1u_2 + b_1h_2(u_1, u_2) + c_1\big(c\phi'(u_2) + bh_2(u_1, u_2)\big) + d_1h_2\big(u_1, c\phi'(u_2) + bh_2(u_1, u_2)\big) = 0. \tag{7.39}$$

As (7.34) - (7.36) are all equivalent, the polynomials in the left sides of (7.38) and (7.39) are, in fact, the zero polynomial (i.e. all the coefficients vanish). Recall the formulas of $h_1(u_1, u_2)$ and $h_2(u_1, u_2)$ given in (7.22) and let

$$\phi'(u_2) = n_1u_2 + n_2u_2^2 + \cdots .$$

We rewrite (7.38) - (7.39) as follows:

$$\begin{aligned}
&\Big(\tau_1u_1 + 2m_{40}u_1^3 + m_{22}u_1u_2^2 + \cdots\Big)\\
-&\Big(\tau_1u_1 + 2m_{40}u_1^3 + m_{22}u_1\big(c(n_1u_2 + n_2u_2^2 + \cdots) + b(\tau_2u_2 + 2m_{04}u_2^3 + m_{22}u_2u_1^2 + \cdots)\big)^2 + \cdots\Big) = 0,
\end{aligned} \tag{7.40}$$

$$\begin{aligned}
&a_1u_2 + b_1\Big(\tau_2u_2 + 2m_{04}u_2^3 + m_{22}u_2u_1^2 + \cdots\Big)\\
&+c_1\Big(c(n_1u_2 + n_2u_2^2 + \cdots) + b(\tau_2u_2 + 2m_{04}u_2^3 + m_{22}u_2u_1^2 + \cdots)\Big)\\
&+d_1\Big(\tau_2\big(c(n_1u_2 + n_2u_2^2 + \cdots) + b(\tau_2u_2 + 2m_{04}u_2^3 + m_{22}u_2u_1^2 + \cdots)\big)\\
&+2m_{04}\big(c(n_1u_2 + n_2u_2^2 + \cdots) + b(\tau_2u_2 + 2m_{04}u_2^3 + m_{22}u_2u_1^2 + \cdots)\big)^3\\
&+m_{22}\big(c(n_1u_2 + n_2u_2^2 + \cdots) + b(\tau_2u_2 + 2m_{04}u_2^3 + m_{22}u_2u_1^2 + \cdots)\big)u_1^2 + \cdots\Big) = 0.
\end{aligned} \tag{7.41}$$

The coefficients of $(u_1u_2^2)$-term in (7.40) and u_2-term in (7.41) are

$$m_{22} - m_{22}(cn_1 + b\tau_2)^2 = 0 \tag{7.42}$$

and

$$a_1 + b_1\tau_2 + c_1(cn_1 + b\tau_2) + d_1\tau_2(cn_1 + b\tau_2) = a_1 + b_1\tau_2 + (c_1 + d_1\tau_2)(cn_1 + b\tau_2) = 0 \tag{7.43}$$

respectively. Since $m_{22} \neq 0$ by the assumption, (7.42) and (7.43) imply either

$$cn_1 + b\tau_2 = 1, \quad a_1 + b_1\tau_2 = -c_1 + d_1\tau_2 \tag{7.44}$$

or

$$cn_1 + b\tau_2 = -1, \quad a_1 + b_1\tau_2 = c_1 + d_1\tau_2. \tag{7.45}$$

As τ_2 is non-real and a_1, b_1, c_1, d_1 are integers, we get

$$a_1 = c_1, \quad b_1 = d_1 \quad \text{or} \quad a_1 = -c_1, \quad b_1 = -d_1. \tag{7.46}$$

Now the coefficient of $(u_1^2u_2)$-term in (7.41) is

$$b_1m_{22} + c_1bm_{22} + d_1\tau_2bm_{22} + d_1m_{22}(cn_1 + b\tau_2) = 0,$$

and so

$$b_1 + c_1 b + d_1\tau_2 b + d_1(cn_1 + b\tau_2) = 0$$

since $m_{22} \neq 0$ by the assumption. Thus, combining with (7.44) - (7.45), we get

$$b_1 + c_1 b + d_1\tau_2 b + d_1 = b_1 - \frac{b_2}{c_2}(c_1 + d_1\tau_2) + d_1 = b_1 - \frac{b_2 c_1}{c_2} + d_1 - \frac{b_2 d_1}{c_2}\tau_2 = 0 \tag{7.47}$$

or

$$b_1 + c_1 b + d_1\tau_2 b - d_1 = b_1 - \frac{b_2}{c_2}(c_1 + d_1\tau_2) - d_1 = b_1 - \frac{b_2 c_1}{c_2} - d_1 - \frac{b_2 d_1}{c_2}\tau_2 = 0. \tag{7.48}$$

Since b_i, c_i, d_i are integers and τ_2 is non-real, both cases imply $b_2 = 0$ or $d_1 = 0$. If $d_1 = 0$, then $b_1 = 0$ by (7.46), and so $c_1 b_2 = 0$ by (7.47) - (7.48). If $c_1 = 0$, then $a_1 = 0$ by (7.46), which is a contradiction (since $a_1 = b_1 = c_1 = d_1 = 0$). So we always fall into $b_2 = 0$.

Suppose $b_2 = 0$ and

$$a_1 = -c_1, \quad b_1 = -d_1 \tag{7.49}$$

or

$$a_1 = c_1, \quad b_1 = d_1. \tag{7.50}$$

Without loss of generality, we consider (7.49). Then (7.38) - (7.39) can be simplified as

$$h_1(u_1, u_2) = h_1\big(u_1, \phi(u_2)\big), \tag{7.51}$$

$$a_1 u_2 + b_1 h_2(u_1, u_2) - a_1\phi(u_2) - b_1 h_2\big(u_1, \phi(u_2)\big) = 0. \tag{7.52}$$

Suppose $\phi(u_2) \neq u_2$ and

$$\phi(u_2) = u_2 + n_i u_2^i + \cdots \tag{7.53}$$

where $n_i \neq 0$. We let $u_1 = 0$ in (7.52), and so

$$\begin{aligned} & a_1 u_2 + b_1 h_2(0, u_2) - a_1\phi(u_2) - b_1 h_2\big(0, \phi(u_2)\big) \\ &= a_1 u_2 + b_1(\tau_2 u_2 + 2m_{04}u_2^3 + \cdots) - a_1(u_2 + n_i u_2^i + \cdots) \qquad (7.54) \\ & -b_1\tau_2\Big((u_2 + n_i u_2^i + \cdots) + 2m_{04}(u_2 + n_i u_2^i + \cdots)^3 + \cdots\Big) \qquad (7.55) \end{aligned}$$

Then the coefficient of u_2^i in (7.55) is

$$-(a_1 + b_1\tau_2)n_i,$$

which is nonzero. But this contradicts the fact that (7.52) is the zero polynomial. So $u_2' = \phi(u_2) = u_2$.

On the other hand, if we consider (7.50), we get $u_2' = \phi(u_2) = -u_2$. $\square$

By the above claim, each $\mathcal{Y}_i$ is contained in an algebraic group defined by

$$M_1 = M_1', \quad L_1 = L_1', \quad , M_2 = M_2', \quad , L_2 = L_2'$$

or

$$M_1 = M_1', \quad L_1 = L_1', \quad , M_2 = (M_2')^{-1}, \quad , L_2 = (L_2')^{-1}.$$

But this contradicts the assumption (7.15).

Part II Now we consider the second case

$$t_1 = t_2', \quad t_2 = t_1'. \tag{7.56}$$

Let

$$P = \Big(t_1^{-q_1}, t_1^{p_1}, t_2^{-q_2}, t_2^{p_2}, (t_1')^{-q_1'}, (t_1')^{p_1'}, (t_2')^{-q_2'}, (t_2')^{p_2'}\Big)$$

and

$$\mathbf{v_1} = (-q_1, p_1, -q_2', p_2'),$$
$$\mathbf{v_2} = (-q_2, p_2, -q_1', p_1').$$

By Siegel's lemma, there exist $\mathbf{b_{11}}, \mathbf{b_{12}}, \mathbf{b_{13}} \in \mathbb{Z}^4$ which vanish at

$$-q_1X_1 + p_1X_2 - q_2'X_3 + p_2'X_4 = 0, \tag{7.57}$$

with $|\mathbf{b_{11}}||\mathbf{b_{12}}||\mathbf{b_{13}}| \leq |\mathbf{v_1}|$, and $\mathbf{b_{21}}, \mathbf{b_{22}}, \mathbf{b_{23}} \in \mathbb{Z}^4$ which vanish at

$$-q_2X_1 + p_2X_2 - q_1'X_3 + p_1'X_4 = 0, \tag{7.58}$$

with $|\mathbf{b_{21}}||\mathbf{b_{22}}||\mathbf{b_{23}}| \leq |\mathbf{v_2}|$. Let

$$\mathbf{b_{11}} = (a_{11}, b_{11}, c_{11}, d_{11}), \quad \mathbf{b_{12}} = (a_{12}, b_{12}, c_{12}, d_{12}),$$
$$\mathbf{b_{21}} = (a_{21}, b_{21}, c_{21}, d_{21}), \quad \mathbf{b_{22}} = (a_{22}, b_{22}, c_{22}, d_{22}),$$

and H be an algebraic subgroup defined by

$$M_1^{a_{11}} L_1^{b_{11}} (M_2')^{c_{11}} (L_2')^{d_{11}} = 1,$$
$$M_1^{a_{12}} L_1^{b_{12}} (M_2')^{c_{12}} (L_2')^{d_{12}} = 1,$$
$$M_2^{a_{21}} L_2^{b_{21}} (M_1')^{c_{21}} (L_1')^{d_{21}} = 1,$$
$$M_2^{a_{22}} L_2^{b_{22}} (M_1')^{c_{22}} (L_1')^{d_{22}} = 1.$$

Following the same argument given in **Part I** (i.e. Claim 7.4), if P is an isolated point of $(\mathcal{X} \times \mathcal{X}) \cap H$, we get the degree of P is uniformly bounded.

Next suppose the component of $(\mathcal{X} \times \mathcal{X}) \cap H$ containing P, denoted by $\mathcal{Y}$, is an anomalous subvariety of $\mathcal{X} \times \mathcal{X}$. First note that, by Lemma 7.2, $\mathcal{Y}$ does not contain z^0. If $\mathcal{X} \times \mathcal{X}$ contains only finitely many anomalous subvarieties near z^0 and $\mathcal{Y}$ is one of them, then, by shrinking the size of a neighborhood of z^0, we exclude those Dehn filling points contained in $\mathcal{Y}$.

Now we assume $\mathcal{X} \times \mathcal{X}$ contains infinitely many anomalous subvarieties near z^0 and each of them contains a Dehn filling point coming from two isometric Dehn filled manifolds of different filling coefficients. More precisely, we consider two infinite sequences of two co-prime pairs $(p_{1i}/q_{1i}, p_{2i}/q_{2i})_{i \in \mathcal{I}}$ and $(p_{1i}'/q_{1i}', p_{2i}'/q_{2i}')_{i \in \mathcal{I}}$ such that, for each $i \in \mathcal{I}$,

$$(p_{1i}/q_{1i}, p_{2i}/q_{2i}) \neq (p_{1i}'/q_{1i}', p_{2i}'/q_{2i}'),$$

and

$$t_{1i} = t_{2i}', \quad t_{2i} = t_{1i}'$$

where $\{t_{1i}, t_{2i}\}$ and $\{t_{1i}', t_{2i}'\}$ are the sets of holonomies of $\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})$ and $\mathcal{M}(p_{1i}'/q_{1i}', p_{2i}'/q_{2i}')$ respectively. Let

$$P_i = \left(t_{1i}^{-q_{1i}}, t_{1i}^{p_{1i}}, t_{2i}^{-q_{2i}}, t_{2i}^{p_{2i}}, (t_{1i}')^{-q_{1i}'}, (t_{1i}')^{p_{1i}'}, (t_{2i}')^{-q_{2i}'}, (t_{2i}')^{p_{2i}'}\right)$$

be the Dehn filling point associated to $\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})$ and $\mathcal{M}(p_{1i}'/q_{1i}', p_{2i}'/q_{2i}')$ in $\mathcal{X} \times \mathcal{X}$, and H_i be an algebraic subgroup containing P_i and obtained by the same procedure shown

earlier (i.e. using Siegel's lemma). Thus H_i is defined by equations of the following form:

$$\begin{aligned} M_1^{a_{1i}}L_1^{b_{1i}}(M_2')^{a_{1i}'}(L_2')^{b_{1i}'} &= 1,\\ M_1^{a_{2i}}L_1^{b_{2i}}(M_2')^{a_{2i}'}(L_2')^{b_{2i}'} &= 1,\\ M_2^{c_{1i}}L_2^{d_{1i}}(M_1')^{c_{1i}'}(L_1')^{d_{1i}'} &= 1,\\ M_2^{c_{2i}}L_2^{d_{2i}}(M_1')^{c_{2i}'}(L_1')^{d_{2i}'} &= 1. \end{aligned} \tag{7.59}$$

We denote the component of $H_i \cap (\mathcal{X} \times \mathcal{X})$ containing P_i by $\mathcal{Y}_i$, and assume $\{\mathcal{Y}_i\}_{i\in\mathcal{I}}$ is a family of infinitely many anomalous subvarieties near z^0. Then, as we checked in Claim 5.4, the component of $H_i \cap (\mathcal{X} \times \mathcal{X})$ containing z^0 is also an anomalous subvariety of $\mathcal{X} \times \mathcal{X}$ for each i. But this is impossible by Lemma 7.2.

Summing up, if (7.7) holds, then either (7.8) holds or the degrees of t_1 and t_2 are bounded. By Theorem 3.11, the heights of t_1 and t_2 are uniformly bounded and, by Northcott's theorem, there are only a finite number of choices for the set of holonomies given in (7.7). Combining with Theorem 2.5, we conclude there are only a finite number of Dehn filling coefficients having the same set of holonomies. Thus, except for those finitely many choices, the only case that makes (7.7) possible is (7.8). This completes the proof. □

References

[1] R. Benedetti, C. Petronio, *Lectures on Hyperbolic Geometry*, Springer (1992).

[2] S. Bleiler, C. Hogson, J. Weeks, *Cosmetic surgery on knots*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr. 2, Geom. Topol. Publ., Coventry (1999), 23-34.

[3] E. Bombieri, D. Masser, U. Zannier, *Intersecting a plane with algebraic subgroups of multiplicative groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2008), 51-80.

[4] E. Bombieri, D. Masser, U. Zannier, *Anomalous subvarieties-structure theorems and applications*, IMRN (2007), 1-33.

[5] E. Bombieri, D. Masser, U. Zannier, *Intersection of complex varieties with tori*, Acta Arith. (2008), 309-323.

[6] E. Bombieri, W. Gubler, *Heights in Diophantine Geometry*, Cambridge University Press (2006).

[7] C. Gordon, *Dehn surgery on knots*, Proceedings of the ICM, Vol. I, II (Kyoto, 1990) Math. Soc. Japan, Tokyo (1991), 631-642.

[8] P. Habegger, *On the bounded height conjecture*, IMRN (2009), 860-886.

[9] B. Jeon, *Hyperbolic 3-manifolds of bounded volume and trace field degree*, arxiv.org/abs/1305.0611.

[10] B. Jeon, *Hyperbolic 3-manifolds of bounded volume and trace field degree II*, arxiv.org/abs/1409.2069.

[11] R. Kirby, *Problems in low-dimensional topology*, AMS/IP Studies in Advanced Mathematics, Vol 2, part 2, Amer. Math. Soc. (1997).

[12] M. Lackenby, *Dehn surgery on knots in 3-manifolds*, J. Amer. Math. Soc. (1997), 835-864.

[13] W. Neumann, A. Reid, *Rigidity of cusps in deformations of hyperbolic 3-orbifolds*, Math. Ann. (1993), 223-237.

[14] W. Neumann, D. Zagier, *Volumes of hyperbolic three-manifolds*, Topology (1985), 307-332.

[15] Y. Ni, Z. Wu, *Cosmetic surgeries on knots in S^3* ,J. Reine Angew. Math. (2015), 1-17.

[16] P. Ozsvàth and Z. Szabò, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. (2011), 1-68.

[17] W. Thurston, *The Geometry and Topology of 3-manifolds*, Princeton University Mimeographed Notes (1979).

[18] J. Wang, *Cosmetic surgeries on genus one knots*, Algebr. Geom. Topol. (2006), 1491-1517.

[19] U. Zannier, *Some problems of unlikely intersections in Arithmetic and Geometry*, Princeton University Press (2012).

Department of Mathematics
Columbia University
2990 Broadway, New York, NY 10027

Email Address: jeon@math.columbia.edu